PARAMETRIZING HITCHIN COMPONENTS

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ABSTRACT. We construct a geometric, real analytic parametrization of the Hitchin component $\mathrm{Hit}_n(S)$ of the $\mathrm{PSL}_n(\mathbb{R})$–character variety $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ of a closed surface S. The approach is explicit and constructive. In essence, our parametrization is an extension of Thurston's shearing coordinates for the Teichmüller space of a closed surface, combined with Fock-Goncharov's coordinates for the moduli space of positive framed local systems of a punctured surface. More precisely, given a maximal geodesic lamination $\lambda \subset S$ with finitely many leaves, our coordinates are of two types, and consist of shear invariants associated with each leaf of λ, and of triangle invariants associated with each component of the complement $S - \lambda$. Besides, we compute and describe various identities and relations between these two invariants.

For a closed, connected, oriented surface S of genus $g > 1$, this article is concerned with the space of homomorphisms $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ from the fundamental group $\pi_1(S)$ to the Lie group $\mathrm{PSL}_n(\mathbb{R})$ (equal to the special linear group $\mathrm{SL}_n(\mathbb{R})$ if n is odd, and to $\mathrm{SL}_n(\mathbb{R})/\pm\mathrm{Id}$ if n is even), and more precisely with a preferred component $\mathrm{Hit}_n(S)$ of the character variety

$$\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S) = \{\text{homomorphisms } \rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})\} /\!\!/ \mathrm{PSL}_n(\mathbb{R}),$$

where the group $\mathrm{PSL}_n(\mathbb{R})$ acts on homomorphisms $\pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ by conjugation. The precise definition of the character variety $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ requires that the quotient be taken in the sense of geometric invariant theory [MFK]; however, for the component $\mathrm{Hit}_n(S)$ that we are interested in, this quotient construction coincides with the usual topological quotient [Hi]. Also, note that the consideration of homomorphisms $\pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ is essentially equivalent, by arguments involving the cohomology groups $H^1(S;\mathbb{R}^*)$ and $H^1(S;\mathbb{Z}_2)$, to the analysis of general representations $\pi_1(S) \to \mathrm{GL}_n(\mathbb{R})$.

In the case where $n = 2$, the character variety $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{R})}(S)$ has $4g-3$ components [Go$_1$]. Two of these components correspond to all injective homomorphisms $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ whose image $\rho\big(\pi_1(S)\big)$ is discrete in $\mathrm{PSL}_2(\mathbb{R})$. Identifying $\mathrm{PSL}_2(\mathbb{R})$ with the orientation-preserving isometry group of the hyperbolic plane $\mathbb{H}^2$, the orientation of S then singles out one of these two components, where the natural equivalence relation $S \to \mathbb{H}^2/\rho\big(\pi_1(S)\big)$ has degree $+1$. This preferred component of $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{R})}(S)$ is the *Teichmüller component* $\mathcal{T}(S)$. By the Uniformization Theorem, the Teichmüller component $\mathcal{T}(S)$ is diffeomorphic to the space of complex structures on S, and consequently plays a fundamental rôle in complex analysis as well as in 2–dimensional hyperbolic geometry. In particular, a classical result is that it is diffeomorphic to $\mathbb{R}^{6(g-1)}$.

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In the general case, there is a preferred homomorphism $\mathrm{PSL_2}(\mathbb{R}) \to \mathrm{PSL_n}(\mathbb{R})$ coming from the unique n–dimensional representation of $\mathrm{SL_2}(\mathbb{R})$ (or, equivalently, from the natural action of $\mathrm{SL_2}(\mathbb{R})$ on the vector space $\mathbb{R}[X,Y]_{n-1} \cong \mathbb{R}^n$ of homogeneous polynomials of degree $n-1$ in two variables). This provides a natural map $\mathcal{R}_{\mathrm{PSL_2}(\mathbb{R})}(S) \to \mathcal{R}_{\mathrm{PSL_n}(\mathbb{R})}(S)$. The *Hitchin component* $\mathrm{Hit_n}(S)$ is the component of $\mathcal{R}_{\mathrm{PSL_n}(\mathbb{R})}(S)$ that contains the image of the Teichmüller component of $\mathcal{R}_{\mathrm{PSL_2}(\mathbb{R})}(S)$. A *Hitchin representation* is a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$ representing an element of the Hitchin component. The terminology is motivated by the following fundamental result of N. Hitchin [Hi], who was the first to single out this component.

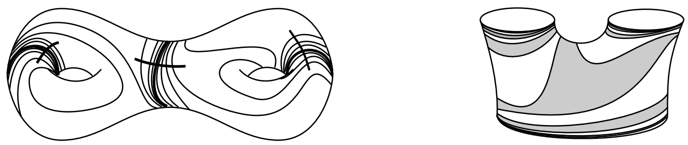
Theorem 1 (Hitchin)**.** *When $n \geqslant 3$, the character variety has 3 or 6 components according to whether n is odd or even, and the Hitchin component $\mathrm{Hit_n}(S)$ is diffeomorphic to $\mathbb{R}^{2(g-1)(n^2-1)}$.*

Hitchin's proof of these results is based on the theory of Higgs bundles, and relies on techniques of geometric analysis. Hitchin notes in [Hi] that these methods do not provide any geometric information on individual Hitchin representations. A few years after Hitchin's work, S. Choi and W. Goldman [ChGo] showed that, in the case where $n = 3$, the Hitchin component $\mathrm{Hit_n}(S)$ parametrizes the deformation space of *real convex projective structures* on S. In particular, using this point of view, Goldman [$\mathrm{Go_2}$] independently provided an explicit parametrization of the Hitchin component $\mathrm{Hit_3}(S)$ by $\mathbb{R}^{16(g-1)}$, via an extenion of the classical Fenchel-Nielsen coordinates for the Teichmüller space $\mathcal{T}(S)$.

A decade later, F. Labourie [La] showed, among other properties, that Hitchin representations are injective and have discrete image in $\mathrm{PSL_n}(\mathbb{R})$. He achieved this by establishing a very powerful Anosov property for Hitchin representations. This Anosov property associates to each Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$ a certain *flag curve* valued in the space $\mathrm{Flag}(\mathbb{R}^n)$ of all flags in $\mathbb{R}^n$, which is invariant under the image $\rho\big(\pi_1(S)\big) \subset \mathrm{PSL_n}(\mathbb{R})$. The same invariant flag curve was similarly provided by independent work of V. Fock and A. Goncharov [FoG], who in addition established a certain positivity condition for this flag curve. This approach also proves the faithfulness and the discreteness of Hitchin representations. The point of view of Fock and Goncharov is algebaic geometric and relies on G. Lusztig's notion of positivity [Lu1, Lu2]; in particular, it is very different from Labourie's.

The main achievement of the current paper is to provide a new parametrization of the Hitchin component $\mathrm{Hit_n}(S)$ by $\mathbb{R}^{2(g-1)(n^2-1)}$, which is much more closely related to the geometry of Hitchin representations than Hitchin's original parametrization. It relies on the methods developed by Labourie and Fock-Goncharov, and is somewhat reminiscent of the classical Fenchel-Nielsen coordinates (see for instance [Hu, §7.6]) for the Teichmüller space $\mathcal{T}(S)$. When $n = 2$, this parametrization coincides with the parametrization of the Teichmüller space via the shear coordinates associated with a maximal geodesic lamination λ that were developed in [$\mathrm{Th_2}$, $\mathrm{Bo_1}$].

We begin with some topological data on the surface S, consisting of a maximal geodesic lamination λ with finitely many leaves. For instance, in the spirit of the Fenchel-Nielsen coordinates, such a geodesic lamination can be obtained from a decomposition of S into pairs of pants, and by cutting each pair of pants along three bi-infinite curves spiraling around the boundary components to obtain three infinite triangles; the geodesic lamination λ then consists of the $3(g-1)$ closed



(a) (b)

FIGURE 1. A finite geodesic lamination coming from a pair of pants decomposition

curves of the pair of pants decomposition, together with the $6(g-1)$ spiraling bi-infinite curves; see Figure 1(a) for an example, while Figure 1(b) illustrates how to split a pair of pants along three spiraling curves to obtain two infinite triangles. In general, for an arbitrary auxiliary metric of negative curvature on S, a maximal geodesic lamination with finitely many leaves consists of s disjoint closed geodesics, with $1 \leqslant s \leqslant 3(g-1)$, and of $6(g-1)$ disjoint bi-infinite geodesics whose ends spiral around these closed geodesics and which split the surface S into $4(g-1)$ infinite triangles.

Given a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$, the Anosov structure discovered by Labourie and the positivity property introduced by Fock-Goncharov enable us to read a certain number of invariants of ρ. These include $\frac{1}{2}(n-1)(n-2)$ real numbers (called *triangle invariants*) associated with each of the $4(g-1)$ triangles of $S - \lambda$, and $n-1$ real numbers (called *shear invariants*) associated with each of the $6(g-1)+s$ leaves of the geodesic lamination λ.

The triangle invariants, and the shear invariants associated with the infinite leaves, were introduced by Fock and Goncharov in their parametrization [FoG] of the so-called moduli space of positive framed local systems of a surface S, where S is required to have at least one puncture. This moduli space is the natural extension of the Hitchin component to punctured surfaces; see [BAG] for the Higgs bundle point of view on this space. The construction of the shear invariants associated with closed leaves is very similar to that of infinite leaves.

A major difference with the punctured-surface case of Fock and Goncharov lies in the fact that, when the surface S is closed, the triangle and shear invariants are not independent of each other. Indeed, they satisfy $n-1$ linear equalities and $n-1$ linear inequalities for each of the $s \geqslant 1$ closed leaves of λ. It turns out that these equalities and inequalities are the only relations satisfied by these invariants, and that they can be used to parametrize $\mathrm{Hit_n}(S)$.

Theorem 2. *The above triangle and shear invariants provide a real-analytic parametrization of the Hitchin component* $\mathrm{Hit_n}(S)$ *by the interior of a convex polytope of dimension* $2(g-1)(n^2-1)$.

In the special case where $n=3$, and where λ is a maximal lamination obtained by adding spiraling bi-infinite curves to a pair of pants decomposition of S as in Figure 1, our parametrization of $\mathrm{Hit_n}(S)$ is similar in spirit to the one developed by Goldman [Go$_2$], but different in its details.

When $n=2$, there are no triangle invariants and, as indicated earlier, the parametrization of Theorem 2 coincides with the parametrization of the Teichmüller space $\mathcal{T}(S)$ by shear coordinates [Th$_2$, Bo$_1$]. In the general case, because $S - \lambda$ consists of $4(g-1)$ triangles, the triangle invariants define a map $\mathrm{Hit_n}(S) \to$

$\mathbb{R}^{2(g-1)(n-1)(n-2)}$. It turns out that there are global linear relations between these triangle invariants:

Proposition 3. *The image of the map* $\mathrm{Hit_n}(S) \to \mathbb{R}^{2(g-1)(n-1)(n-2)}$ *defined by triangle invariants is contained in a linear subspace of codimension* $\lfloor \frac{1}{2}(n-1) \rfloor$ *in* $\mathbb{R}^{2(g-1)(n-1)(n-2)}$ *(where* $\lfloor x \rfloor$ *denotes the largest integer* $\leqslant x$*).*

The existence of constraints for the triangle invariants were somewhat unexpected to us. They can be explained by a more conceptual approach that uses the length functions of [Dr], combined with a homological argument; see [BoD]. In fact, the abstract proof of [BoD] preceded the explicit computational argument that we give in the current article.

1. Generic triples and quadruples of flags

The construction of our invariants of Hitchin representations heavily relies on finite collections of flags in $\mathbb{R}^n$.

1.1. **Flags.** A *flag* in $\mathbb{R}^n$ is a family F of nested linear subspaces $F^{(0)} \subset F^{(1)} \subset \cdots \subset F^{(n-1)} \subset F^{(n)}$ of $\mathbb{R}^n$ where each $F^{(a)}$ has dimension a.

A pair of flags (E, F) is *generic* if every subspace $E^{(a)}$ of E is transverse to every subspace $F^{(b)}$ of F. This is equivalent to the property that $E^{(a)} \cap F^{(n-a)} = 0$ for every a.

Similarly, a triple of flags (E, F, G) is *generic* if each triple of subspaces $E^{(a)}$, $F^{(b)}$, $G^{(c)}$, respectively in E, F, G, meets transversely. Again, this is equivalent to the property that $E^{(a)} \cap F^{(b)} \cap G^{(c)} = 0$ for every a, b, c with $a + b + c = n$.

1.2. **Triple ratios of generic flag triples.** Elementary linear algebra shows that, for any two generic flag pairs (E, F) and (E', F'), there is a linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$ sending E to E' and F to F'. However, the same is not true for generic flag triples. Indeed, there is a whole moduli space of generic flag triples modulo the action of $\mathrm{PSL_n}(\mathbb{R})$, and this moduli space can be parametrized by invariants that we now describe. These invariants are expressed in terms of the exterior algebra $\Lambda(\mathbb{R}^n)$ of $\mathbb{R}^n$.

Let (E, F, G) be a generic flag triple. For each a, b, c between 0 and n, the spaces $\Lambda^a\big(E^{(a)}\big)$, $\Lambda^b\big(F^{(b)}\big)$ and $\Lambda^c\big(G^{(c)}\big)$ are all isomorphic to $\mathbb{R}$. Choose non-zero elements $e^{(a)} \in \Lambda^a\big(E^{(a)}\big)$, $f^{(b)} \in \Lambda^b\big(F^{(b)}\big)$ and $g^{(c)} \in \Lambda^c\big(G^{(c)}\big)$. We will use the same letters to denote their images $e^{(a)} \in \Lambda^a(\mathbb{R}^n)$, $f^{(b)} \in \Lambda^b(\mathbb{R}^n)$ and $g^{(c)} \in \Lambda^c(\mathbb{R}^n)$.

Given integers a, b, $c \geqslant 1$ with $a + b + c = n$, we then define the (a, b, c)*–triple ratio* of the generic flag triple (E, F, G) as the number

$$T_{abc}(E,F,G) = \frac{e^{(a+1)} \wedge f^{(b)} \wedge g^{(c-1)}}{e^{(a-1)} \wedge f^{(b)} \wedge g^{(c+1)}} \frac{e^{(a)} \wedge f^{(b-1)} \wedge g^{(c+1)}}{e^{(a)} \wedge f^{(b+1)} \wedge g^{(c-1)}} \frac{e^{(a-1)} \wedge f^{(b+1)} \wedge g^{(c)}}{e^{(a+1)} \wedge f^{(b-1)} \wedge g^{(c)}}$$

where each of the six wedge products are elements of $\Lambda^n(\mathbb{R}^n) \cong \mathbb{R}$. The fact that the flag triple (E, F, G) is generic guarantees that these wedge products are non-zero, so that the three ratios make sense. Also, because all the spaces $\Lambda^{a'}\big(E^{(a')}\big)$, $\Lambda^{b'}\big(F^{(b')}\big)$ and $\Lambda^{c'}\big(G^{(c')}\big)$ involved in the expression are isomorphic to $\mathbb{R}$, this triple ratio is independent of the choice of the non-zero elements $e^{(a')} \in \Lambda^{a'}\big(E^{(a')}\big)$,

$f^{(b')} \in \Lambda^{b'}\left(F^{(b')}\right)$ and $g^{(c')} \in \Lambda^{c'}\left(G^{(c')}\right)$; indeed, each of these elements appears twice in the expression, once in a numerator and once in a denominator.

The natural action of the linear group $\mathrm{GL_n}(\mathbb{R})$ on the flag variety $\mathrm{Flag}(\mathbb{R}^n)$ descends to an action of the projective linear group $\mathrm{PGL_n}(\mathbb{R})$, quotient of $\mathrm{GL_n}(\mathbb{R})$ by its center $\mathbb{R}^*\mathrm{Id}$ consisting of all non-zero scalar multiples of the identity. Note that the projective special linear group $\mathrm{PSL_n}(\mathbb{R})$ is equal to $\mathrm{PGL_n}(\mathbb{R})$ if n is odd, and is an index 2 subgroup of $\mathrm{PGL_n}(\mathbb{R})$ otherwise.

Proposition 4. *Two generic flag triples (E, F, G) and (E', F', G') are equivalent under the action of $\mathrm{PGL_n}(\mathbb{R})$ if and only if $T_{abc}(E, F, G) = T_{abc}(E', F', G')$ for every a, b, $c \geqslant 1$ with $a + b + c = n$.*

In addition, for any set of non-zero numbers $t_{abc} \in \mathbb{R}^*$, there exists a generic flag triple (E, F, G) such that $T_{abc}(E, F, G) = t_{abc}$ for every a, b, $c \geqslant 1$ with $a+b+c = n$.

Proof. See [FoG, §9]. □

Note the elementary property of triple ratios under permutation of the flags.

Lemma 5.

$$T_{abc}(E, F, G) = T_{bca}(F, G, E) = T_{bac}(F, E, G)^{-1}. \qquad \square$$

1.3. **Quadruple ratios of generic flag triples.** In addition to triple ratios, a similar type of invariants of generic flag triples will play an important rôle in our analysis of Hitchin representations.

For an integer a with $1 \leqslant a \leqslant n-1$, the *$a$–th quadruple ratio* of the generic flag triple (E, F, G) is the number

$$Q_a(E, F, G) = \frac{e^{(a-1)} \wedge f^{(n-a)} \wedge g^{(1)}}{e^{(a)} \wedge f^{(n-a-1)} \wedge g^{(1)}} \, \frac{e^{(a)} \wedge f^{(1)} \wedge g^{(n-a-1)}}{e^{(a-1)} \wedge f^{(1)} \wedge g^{(n-a)}} \, \frac{e^{(a+1)} \wedge f^{(n-a-1)}}{e^{(a+1)} \wedge g^{(n-a-1)}} \, \frac{e^{(a)} \wedge g^{(n-a)}}{e^{(a)} \wedge f^{(n-a)}}$$

where, as before, we consider arbitrary non-zero elements $e^{(a')} \in \Lambda^{a'}\left(E^{(a')}\right)$, $f^{(b')} \in \Lambda^{b'}\left(F^{(b')}\right)$ and $g^{(c')} \in \Lambda^{c'}\left(G^{(c')}\right)$, and where the ratios are computed in $\Lambda^n(\mathbb{R}^n) \cong \mathbb{R}$. As with triple ratios, the number $Q_a(E, F, G) \in \mathbb{R}^*$ is well-defined, independent of choices, and invariant under the action of $\mathrm{PGL_n}(\mathbb{R})$ on the set of generic flag triples.

Note that $Q_a(E, G, F)) = Q_a(E, F, G)^{-1}$, but that this quadruple ratio usually does not behave well under the other permutations of the flags E, F and G, as E plays a special rôle in $Q_a(E, F, G)$.

By Proposition 4, a generic flag triple (E, F, G) is completely determined by its triple ratios modulo the action of the linear group $\mathrm{GL_n}(\mathbb{R})$. It is therefore natural to expect that the quadruple ratio can be expressed in terms of the triple ratios of (E, F, G). This is indeed the case, and the corresponding expression is particularly simple.

Lemma 6. *For $a = 1, 2, \ldots, n-1$,*

$$Q_a(E, F, G) = \prod_{b+c=n-a} T_{abc}(E, F, G)$$

where the product is over all integers b, $c \geqslant 1$ with $b + c = n - a$. In particular, $Q_{n-1}(E, F, G) = 1$ and $Q_{n-2}(E, F, G) = T_{(n-2)11}(E, F, G)$.

Proof. When computing the right-hand side of the equation, most terms $e^{(a')} \wedge f^{(b')} \wedge g^{(c')}$ cancel out and we are left with the eight terms of $Q_a(E, F, G)$. □

1.4. **Double ratios of generic flag quadruples.** We now consider quadruples (E, F, G, G') of flags $E,\ F,\ G,\ G' \in \mathrm{Flag}(\mathbb{R}^n)$. Such a flag quadruple is *generic* if each quadruple of subspaces $E^{(a)},\ F^{(b)},\ G^{(c)},\ G'^{(d)}$ meets transversely. As usual, we can restrict attention to the cases where $a + b + c + d = n$.

For $1 \leqslant a \leqslant n-1$, the *a–th double ratio* of the generic flag quadruple (E, F, G, G') is

$$D_a(E, F, G, G') = -\frac{e^{(a)} \wedge f^{(n-a-1)} \wedge g^{(1)}}{e^{(a)} \wedge f^{(n-a-1)} \wedge g'^{(1)}} \, \frac{e^{(a-1)} \wedge f^{(n-a)} \wedge g'^{(1)}}{e^{(a-1)} \wedge f^{(n-a)} \wedge g^{(1)}}$$

where we choose arbitrary non-zero elements $e^{(a')} \in \Lambda^{a'}(E^{(a')})$, $f^{(b')} \in \Lambda^{b'}(F^{(b')})$, $g^{(1)} \in \Lambda^1(G^{(1)})$ and $g'^{(1)} \in \Lambda^1(G'^{(1)})$. As usual, $D_a(E, F, G, G')$ is independent of these choices. The minus sign is motivated by the notion of positivity that is described in §1.5 and plays a very important rôle in this article (see Proposition 10).

Lemma 7.

$$D_a(E, F, G', G) = D_a(E, F, G, G')^{-1}$$
$$\textit{and } D_a(F, E, G, G') = D_{n-a}(E, F, G, G')^{-1}. \qquad \square$$

1.5. **Positivity.** A flag triple (E, F, G) is *positive* if it is generic and if all its triple ratios $T_{abc}(E, F, G)$ are positive. By Proposition 4, positive flag triples form a component in the space of all generic flag triples. Lemma 5 also shows that positivity of the triple (E, F, G) is preserved under all permutations of the flags E, F and $G \in \mathrm{Flag}(\mathbb{R}^n)$.

A generic flag quadruple (E, F, G, G') is *positive* if it is generic, if the two triples (E, F, G) and (E, F, G') are positive, and if all double ratios $D_a(E, F, G, G')$ are positive.

See [FoG, Lu2, Lu1] for a more conceptual and general definition of positivity, valid for k–tuples of (i.e. partial) flags.

2. Invariants of Hitchin representations

We now define several invariants of Hitchin representations $\rho \colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$. These invariants require that we are given a certain topological information on the surface.

2.1. **The topological data.** Since we are going to use the terminology of geodesic laminations, it is convenient to endow the surface S with a riemannian metric of negative curvature. However, it is well-known that geodesic laminations can also be defined in a metric independent way, and in particular are purely topological objects. See for instance [Th_1, PeH, Bo_2].

Let λ be a maximal geodesic lamination with finitely many leaves. Namely λ is the union of finitely many disjoint simple closed geodesics $c_1, c_2, \ldots, c_s$ and of finitely disjoint many bi-infinite geodesics $g_1, g_2, \ldots, g_t$ in the complement of the c_i, in such a way that each end of a g_j spirals along some c_i, and that the complement $S - \lambda$ consists of finitely many infinite triangles $T_1, T_2, \ldots, T_u$.

An Euler characteristic argument shows that, if g is the genus of the surface S, then the number u of components of $S - \lambda$ is equal to $4(g - 1)$, while the number t of infinite leaves of λ is equal to $6(g - 1)$. The number s of closed leaves of λ can be any integer between 1 and $3(g - 1)$.

For instance, λ can be obtained from a family of disjoint simple closed curves $c_1, c_2, \ldots, c_{3g-3}$ decomposing S into pairs of pants, and then by decomposing each

pair of pants into 2 infinite triangles along 3 infinite geodesics spiraling around the boundary. Figure 1 describes one such example associated with a pair of pants decomposition, and Figure 2 shows another example with only one closed leaf.

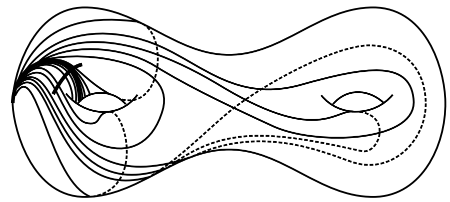


FIGURE 2. A finite geodesic lamination with exactly one closed leaf

We need more data, in addition to the finite-leaved maximal geodesic lamination λ. One is the choice of an orientation on each leaf of λ. This choice is completely free and arbitrary. In particular, we are not making any assumption of continuity regarding these orientations, or of a relationship between these orientations and the directions in which infinite leaves spiral around closed leaves.

Finally, for each closed leaf c_i, we choose an arc k_i that is transverse to λ, cuts c_i in exactly one point, and meets no other closed leaf c_j.

For the reader who is familiar with the case where $n = 2$, we can indicate that the choice of orientations for the leaves of λ is irrelevant in that case. Regarding the need for the transverse arcs k_i, it corresponds to a well-known technical difficulty in the definition of the Fenchel-Nielsen coordinates: the twist parameters are relatively easy to define modulo the length parameters, but require more cumbersome topological information to be well-defined as real numbers.

2.2. **The flag curve of a Hitchin representation.** Once we are given this topological data, the key tool for the construction of our invariants is the Anosov structure for Hitchin representations discovered by F. Labourie [La]. Another good reference for Labourie's work is [Gui].

We begin with what is actually a corollary of these results.

Proposition 8 (Labourie)**.** *Let $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$ be a Hitchin representation. Then, for every non-trivial $\gamma \in \pi_1(S)$, the element $\rho(\gamma) \in \mathrm{PSL_n}(\mathbb{R})$ has real eigenvalues and their absolute values are distinct.* □

When n is even, the eigenvalues of $\rho(\gamma) \in \mathrm{PSL_n}(\mathbb{R}) = \mathrm{SL_n}(\mathbb{R})/\{\pm \mathrm{Id}\}$ are only defined up to sign. However, we can be a little more specific.

Lemma 9. *Let $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$ be a Hitchin representation. Then, for every non-trivial $\gamma \in \pi_1(S)$, the element $\rho(\gamma) \in \mathrm{PSL_n}(\mathbb{R})$ admits a lift $\rho(\gamma)' \in \mathrm{SL_n}(\mathbb{R})$ whose eigenvalues are distinct and all positive.*

Proof. Let the non-trivial element $\gamma \in \pi_1(S)$ be fixed.

The property "all eigenvalues of a lift $\rho(\gamma)' \in \mathrm{SL_n}(\mathbb{R})$ of $\rho(\gamma) \in \mathrm{PSL_n}(\mathbb{R})$ have the same sign" is open and closed in the space of Hitchin representations, since these eigenvalues are real and non-zero. This property holds in the special case where ρ is the composition of a Teichmüller representation $\rho_2\colon \pi_1(S) \to \mathrm{PSL_2}(\mathbb{R})$ with the natural embedding $\mathrm{PSL_2}(\mathbb{R}) \to \mathrm{PSL_n}(\mathbb{R})$; indeed, if $\rho_2(\gamma)$ has a lift $\rho_2(\gamma)' \in \mathrm{SL_2}(\mathbb{R})$ with eigenvalues a and a^{-1}, then $\rho(\gamma)$ has a lift $\rho(\gamma)' \in \mathrm{SL_n}(\mathbb{R})$ with eigenvalues

a^{n-2k+1} as k ranges over all integers with $1 \leqslant k \leqslant n$. Therefore, the property holds for every Hitchin representation by connectedness of the space of Hitchin representations.

This proves that the eigenvalues of any lift $\rho(\gamma)' \in \mathrm{SL_n}(\mathbb{R})$ of $\rho(\gamma) \in \mathrm{PSL_n}(\mathbb{R})$ have the same sign. If these eigenvalues are all negative, note that n is even since $\rho(\gamma)'$ has determinant $+1$. Then $-\rho(\gamma)' \in \mathrm{SL_n}(\mathbb{R})$ is another lift of $\rho(\gamma) \in \mathrm{PSL_n}(\mathbb{R})$, whose eigenvalues are all positive (and distinct by Proposition 8). □

If ρ is a Hitchin representation and if $\gamma \in \pi_1(S)$ is non-trivial, let $\rho(\gamma)' \in \mathrm{SL_n}(\mathbb{R})$ be the lift of $\rho(\gamma) \in \mathrm{PSL_n}(\mathbb{R})$ given by Lemma 9. Let

$$m_1^\rho(\gamma) > m_2^\rho(\gamma) > \cdots > m_n^\rho(\gamma) > 0$$

be the eigenvalues of $\rho(\gamma)'$, indexed in decreasing order. Since these eigenvalues are distinct, $\rho(\gamma)'$ is diagonalizable. Let L_a be the (1–dimensional) eigenspace corresponding to the eigenvalue $m_a^\rho(\gamma)$.

This associates to $\rho(\gamma)$ two preferred flags E, $F \in \mathrm{Flag}(\mathbb{R}^n)$ defined by the property that

$$E^{(a)} = \bigoplus_{b=1}^{a} L_b \text{ and } F^{(a)} = \bigoplus_{b=n-a+1}^{n} L_b.$$

By definition, $E \in \mathrm{Flag}(\mathbb{R}^n)$ is the *stable flag* of $\rho(\gamma) \in \mathrm{PSL_n}(\mathbb{R})$, and F is its *unstable flag*.

Let $\widetilde{S}$ be the universal covering of the surface S, and let $\partial_\infty \widetilde{S}$ be its circle at infinity. Recall that every non-trivial $\gamma \in \pi_1(S)$ fixes two points of $\partial_\infty \widetilde{S}$, one of them attracting and the other one repelling.

Proposition 10 (Labourie, Fock-Goncharov)**.** *Given a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$, there exists a unique continuous map $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ such that:*

1. *if $x \in \partial_\infty \widetilde{S}$ is the attracting fixed point of $\gamma \in \pi_1(S)$, then $\mathcal{F}_\rho(x) \in \mathrm{Flag}(\mathbb{R}^n)$ is the stable flag of $\rho(\gamma) \in \mathrm{PSL_n}(\mathbb{R})$;*
2. *$\mathcal{F}_\rho$ is equivariant with respect to the Hitchin homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$, in the sense that $\mathcal{F}_\rho(\gamma x) = \rho(\gamma)(x)$ for every $\gamma \in \pi_1(S)$ and every $x \in \partial_\infty \widetilde{S}$;*
3. *for any two distinct points x, $y \in \partial_\infty \widetilde{S}$, the flag pair $\big(\mathcal{F}_\rho(x), \mathcal{F}_\rho(y)\big)$ is generic;*
4. *for any three distinct points x, y, $z \in \partial_\infty \widetilde{S}$, the flag triple $\big(\mathcal{F}_\rho(x), \mathcal{F}_\rho(y), \mathcal{F}_\rho(z)\big)$ is positive;*
5. *for any four distinct points x, y, z, z' occurring in this order around the circle at infinity $\partial_\infty \widetilde{S}$, the flag quadruple $\big(\mathcal{F}_\rho(x), \mathcal{F}_\rho(y), \mathcal{F}_\rho(z), \mathcal{F}_\rho(z')\big)$ is positive.* □

By definition, this curve $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ is the *flag curve* of the Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$.

The first three properties of Proposition 10 are immediate consequences of the Anosov structure of [La]. The positivity properties of the last two conditions of Proposition 10 were proved by Fock and Goncharov [FoG]; see also the hyperconvexity property of [La, Gui].

2.3. **Invariants of triangles.** Given a finite maximal geodesic lamination λ as in §2.1 and a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$, the first set of invariants of ρ is associated with the components of the complement $S - \lambda$. Recall that each of these components is an ideal triangle.

Consider such a triangle T_j, and select one of its vertices v_j. (Such a vertex is of course not an actual point of the surface S; we let the reader devise a formal definition for a vertex of the ideal triangle $T_j \subset S$.) Lift T_j to to an ideal triangle $\widetilde{T}_j$ in the universal covering $\widetilde{S}$, and let $\widetilde{v}_j \in \partial_\infty \widetilde{S}$ be the vertex of $\widetilde{T}_j$ corresponding to the vertex v_j of T_j. Label the vertices of $\widetilde{T}_j$ as $\widetilde{v}_j$, $\widetilde{v}'_j$ and $\widetilde{v}''_j \in \partial_\infty \widetilde{S}$ in clockwise order around $\widetilde{T}_j$. We can then consider the flag triple $\big(\mathcal{F}_\rho(\widetilde{v}_j), \mathcal{F}_\rho(\widetilde{v}'_j), \mathcal{F}_\rho(\widetilde{v}''_j)\big)$, which is positive by Proposition 10.

We can then consider the (positive) triple ratios of this positive flag triple, and their logarithms

$$\tau^\rho_{abc}(T_j, v_j) = \log T_{abc}\big(\mathcal{F}_\rho(\widetilde{v}_j), \mathcal{F}_\rho(\widetilde{v}'_j), \mathcal{F}_\rho(\widetilde{v}''_j)\big)$$

defined for every a, b, $c \geqslant 1$ with $a+b+c=n$. By ρ–equivariance of the flag curve $\mathcal{F}_\rho$, these triple ratio logarithms depend only on the triangle T_j and on the vertex v_j of T_j, and not on the choice of the lift $\widetilde{T}_j$.

Lemma 5 indicates how the invariant $\tau^\rho_{abc}(T_i, v_j) \in \mathbb{R}$ changes if we choose a different vertex of the triangle T_j.

Lemma 11. *If v_j, v'_j and v''_j are the vertices of T_j, indexed clockwise around T_j, then*

$$\tau^\rho_{abc}(T_i, v_j) = \tau^\rho_{bca}(T_i, v'_j) = \tau^\rho_{cab}(T_i, v''_j). \qquad \square$$

2.4. **Shear invariants of infinite leaves.** Let g_j be an infinite leaf of λ.

Lift g_j to a leaf $\widetilde{g}_j$ of the preimage $\widetilde{\lambda}$ of λ in the universal covering $\widetilde{S}$. This leaf is isolated in $\widetilde{\lambda}$, and is adjacent to two components $\widetilde{T}$ and $\widetilde{T}'$ of the complement $\widetilde{S} - \widetilde{\lambda}$. Choose the notation so that $\widetilde{T}$ and $\widetilde{T}'$ are respectively to the left and to the right of $\widetilde{g}_j$ for the orientation of $\widetilde{g}_j$ coming from the orientation of g_j.

Let x and $y \in \partial_\infty \widetilde{S}$ be the positive and negative end points of $\widetilde{g}_j$, respectively, of $\widetilde{g}_j$. Let z, $z' \in \partial_\infty \widetilde{S}$ be the third vertices of $\widetilde{T}$ and $\widetilde{T}'$, respectively, namely the vertices of these triangles that are neither x nor y. See Figure 3. Consider the flags $E = \mathcal{F}_\rho(x)$, $F = \mathcal{F}_\rho(y)$, $G = \mathcal{F}_\rho(z)$ and $G' = \mathcal{F}_\rho(z')$ associated with these vertices by the flag curve $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$.

For $1 \leqslant a \leqslant n-1$, we can now consider the double ratio $D_a(E, F, G, G')$ as in §1.4. This double ratio is positive by Proposition 10. The *a–th shear invariant* of the Hitchin homomorphism ρ along the oriented leaf g_j is then defined as the logarithm

$$\sigma^\rho_a(g_j) = \log D_a(E, F, G, G').$$

This invariant $\sigma_a(g_j)$ is clearly independent of the choice of the lift $\widetilde{g}_j$ of the leaf g_j to $\widetilde{S}$.

By Lemma 7, reversing the orientation of g_j replaces $\sigma^\rho_a(g_j)$ by $\sigma^\rho_{n-a}(g_j)$.

2.5. **Shear invariants of closed leaves.** The shear invariants of a closed leaf c_i are defined in very much the same way as for infinite leaves, except that we need to use the transverse arc k_i that is part of the topological data to single out two triangles T and T' that are located on either side of c_i.

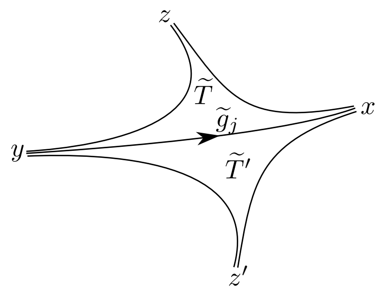


FIGURE 3. The construction of shear invariants of infinite leaves

More precisely, let $\widetilde{c}_i$ be a component of the pre-image of c_i in the universal cover $\widetilde{S}$, and orient it by the orientation of c_i. Lift the arc k_i to an arc $\widetilde{k}_i$ that meets $\widetilde{c}_i$ in one point. Let $\widetilde{T}$ and $\widetilde{T}'$ be the two triangle components of $\widetilde{S} - \widetilde{\lambda}$ that contain the end points of $\widetilde{k}_i$, in such a way that $\widetilde{T}$ and $\widetilde{T}'$ are respectively to the left and to the right of $\widetilde{c}_i$ for the orientation of $\widetilde{c}_i$ lifting the orientation of c_i.

Let x and $y \in \partial_\infty \widetilde{S}$ be the positive and negative end points of $\widetilde{c}_i$, respectively. Among the vertices of $\widetilde{T}$, let $z \in \partial_\infty \widetilde{S}$ be the one that is farthest away from $\widetilde{c}_i$; namely, z is adjacent to the two components of $\widetilde{S} - \widetilde{T}$ that do not contain $\widetilde{c}_i$. Similarly, let z' be the vertex of T' that is farthest away from $\widetilde{c}_i$. See Figures 4(a) and (b) for two of the four possible configurations, according to the directions of the spiraling of infinite leaves around c_i.

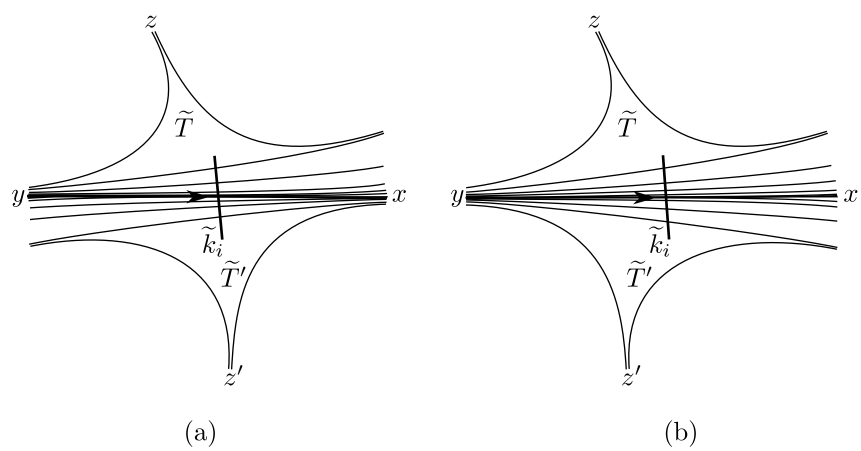


FIGURE 4. The construction of shear invariants of closed leaves

Finally, let $E = \mathcal{F}_\rho(x)$, $F = \mathcal{F}_\rho(y)$, $G = \mathcal{F}_\rho(z)$ and $G' = \mathcal{F}_\rho(z')$ be the flags associated to these vertices by the flag curve $\mathcal{F}_\rho \colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$.

With this data, we again consider for $1 \leqslant a \leqslant n-1$ the double ratio $D_a(E, F, G, G') > 0$ as in §1.4 and Proposition 10. The *a–th shear invariant* of the Hitchin homomorphism ρ along the oriented closed leaf c_i is defined as the logarithm

$$\sigma_a^\rho(c_i) = \log D_a(E, F, G, G').$$

By ρ–equivariance of the flag curve $\mathcal{F}_\rho$, this shear invariant $\sigma_a(c_i)$ is clearly independent of the choice of the component $\widetilde{c}_i$ of the preimage of c_i, and of the lift $\widetilde{k}_i$ of the arc k_i.

Again, Lemma 7 shows that reversing the orientation of c_i replaces $\sigma_a^\rho(c_i)$ by $\sigma_{n-a}^\rho(c_i)$.

2.6. **Lengths of closed leaves.** There are simpler invariants that we could have considered.

Let c be an oriented closed curve in S that is not homotopic to 0, and let $[c] \in \pi_1(S)$ be represented by c after connecting this curve to the base point by an arbitrary path.

If ρ is a Hitchin representation, Lemma 9 asserts that $\rho\big([c]\big) \in \mathrm{PSL_n}(\mathbb{R})$ admits a lift $\rho\big([c]\big)' \in \mathrm{SL_n}(\mathbb{R})$ with distinct real eigenvalues

$$m_1^\rho(c) > m_2^\rho(c) > \cdots > m_n^\rho(c) > 0.$$

For $1 \leqslant a \leqslant n-1$, we define the *$a$–th ρ–length* $\ell_a^\rho(c)$ of the closed curve c for the Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$ as

$$\ell_a^\rho(c) = \log \frac{m_a^\rho(c)}{m_{a+1}^\rho(c)} > 0.$$

Note that this quantity is independent of the class $[c] \in \pi_1(S)$ represented by c, and that reversing the orientation of c replaces $\ell_a^\rho(c)$ by $\ell_{n-a}^\rho(c)$.

The ρ–lengths $\ell_a^\rho(c_i)$ of the closed leaves c_i of the geodesic lamination λ will play an important rôle in the next sections.

Remark 12. The reader should beware of a discrepancy between the conventions of this paper and those of [Dr]: What we call here $\ell_a^\rho(c)$ is called $\ell_a^\rho(c) - \ell_{a+1}^\rho(c)$ in [Dr]. There are two reasons for this change in conventions. The main one is that, as we will see in §3, the ρ–lengths $\ell_a^\rho(c_i)$ of the closed leaves of λ are related to the triangle and shear invariants of ρ, and the expression of this connection is simpler with the current definitions. The second reason comes from the case $n=2$, where the representation ρ defines a hyperbolic metric m on the surface S; then there is exactly one length function and this ρ–length $\ell_1^\rho(c)$ is exactly the classical length of the m–geodesic that is homotopic to c, which plays a fundamental rôle in much of hyperbolic geometry.

3. Relations between invariants

Let c_i be a closed leaf of the geodesic lamination λ. We will express the ρ–lengths $\ell_a(c_i)$ in terms of the triangle invariants $\tau_{abc}^\rho(T_j, v_j)$ and of the shear invariants $\sigma_{n-a}^\rho(g_j)$ of the infinite leaves g_j.

The closed leaf c_i has two sides c_i^{right} or c_i^{left}, respectively located to the left and right of c_i for the chosen orientation of this curve.

Select one of these sides c_i^{right} or c_i^{left}. Let $g_{i_1}, g_{i_2}, \ldots, g_{i_k}$ be the infinite leaves of λ that spiral on this side of c_i. An infinite leaf g_j will appear twice in this list if its two ends spiral on the selected side of c_i. We can then consider the shear parameters $\sigma_a^\rho(g_{i_l}) \in \mathbb{R}$.

Similarly, let $T_{j_1}, T_{j_2}, \ldots, T_{j_k}$ be the components of the complement $S - \lambda$ that spiral on the selected side c_i^{right} or c_i^{left} of c_i. The spiraling of T_{j_l} around this side occurs in the direction of a vertex v_l of T_{j_l}. We can then consider the triangle invariants $\tau_{abc}(T_{j_l}, v_l)$, as in §2.3.

We have to worry about orientations, and more precisely about two types of orientation. One is the orientation of each spiraling leaf g_{i_l}. The other is whether the spiraling occurs in the direction of the orientation of c_i or not.

Proposition 13. *Select a side c_i^{right} or c_i^{left} of the closed leaf c_i of λ. Let g_{i_l} and T_{j_l}, $l=1, 2, \dots, k$ be the infinite leaves and triangles that spiral on this side of c_i. Let v_l be the vertex of the triangle T_{j_l} in the direction of which the spiraling occurs, and consider the triangle invariants $\kappa_a^\rho(T_{j_l}, v_l)$ and the shear invariants $\sigma_a^\rho(g_{i_l})$. Set $\overline{\sigma}_a^\rho(g_{i_l}) = \sigma_a^\rho(g_{i_l})$ if the leaf g_{i_l} is oriented towards c_i, and $\overline{\sigma}_a^\rho(g_{i_l}) = \sigma_{n-a}^\rho(g_{i_l})$ if it is oriented away from c_i. Then:*

(1) *if the selected side is the right-hand side c_i^{right} and if the spiraling occurs in the direction of the orientation of c_i,*
$$\ell_a^\rho(c_i) = \sum_{l=1}^k \overline{\sigma}_a^\rho(g_{i_l}) + \sum_{l=1}^k \sum_{b+c=n-a} \tau_{abc}^\rho(T_{j_l}, v_l);$$
(2) *if the selected side is the right-hand side c_i^{right} and if the spiraling occurs in the direction opposite to the orientation of c_i,*
$$\ell_a^\rho(c_i) = -\sum_{l=1}^k \overline{\sigma}_{n-a}^\rho(g_{i_l}) - \sum_{l=1}^k \sum_{b+c=a} \tau_{(n-a)bc}^\rho(T_{j_l}, v_l);$$
(3) *if the selected side is the left-hand side c_i^{left} and if the spiraling occurs in the direction of the orientation of c_i,*
$$\ell_a^\rho(c_i) = -\sum_{l=1}^k \overline{\sigma}_a^\rho(g_{i_l}) - \sum_{l=1}^k \sum_{b+c=n-a} \tau_{abc}^\rho(T_{j_l}, v_l);$$
(4) *if the selected side is the left-hand side c_i^{left} and if the spiraling occurs in the direction opposite to the orientation of c_i,*
$$\ell_a^\rho(c_i) = \sum_{l=1}^k \overline{\sigma}_{n-a}^\rho(g_{i_l}) + \sum_{l=1}^k \sum_{b+c=a} \tau_{(n-a)bc}^\rho(T_{j_l}, v_l).$$

Proof. First consider the case where the selected side is the right-hand side c_i^{right}, and where the spiraling occurs in the direction of the orientation of c_i.

Without loss of generality, we can assume that the infinite leaves $g_{i_1}, g_{i_2}, \dots, g_{i_k}, g_{i_{k+1}} = g_{i_1}$ and triangles $T_{j_1}, T_{j_2}, \dots, T_{j_k}, T_{j_{k+1}} = T_{j_1}$ are indexed so that the spiraling part of T_{j_l} is bounded on the left by $g_{i_{l-1}}$ and on the right by g_{i_l}.

In the universal covering $\widetilde{S}$, lift the leaves g_{i_l} to leaves $\widetilde{g}_{i_l}$ of the preimage $\widetilde{\lambda}$ of λ, and the triangles T_{j_l} to triangles $\widetilde{T}_{j_l}$, in such a way that $\widetilde{g}_{i_{l-1}}$ and $\widetilde{g}_{i_l}$ are two of the sides of $\widetilde{T}_{j_l}$. See Figure 5. Then, because of the orientation choice for c_i, we have that $[c_i](\widetilde{g}_{i_1}) = \widetilde{g}_{i_{k+1}}$ and $[c_i](\widetilde{T}_{j_1}) = \widetilde{T}_{j_{k+1}}$ for a suitable class $[c_i] \in \pi_1(S)$ represented by the curve c_i.

Let $E = \mathcal{F}_\rho(x) \in \mathrm{Flag}(\mathbb{R}^n)$ be the flag associated by the flag curve $\mathcal{F}_\rho$ to the common end point $x \in \partial_\infty \widetilde{S}$ of the $\widetilde{g}_{i_l}$. Similarly, let $F_l = \mathcal{F}_\rho(x_l)$ be the flag associated with the other end point $x_l \neq x$ of $\widetilde{g}_{i_l}$. It is also convenient to set $G_l = F_{l-1} \in \mathrm{Flag}(\mathbb{R}^n)$.

Pick non-zero elements $e^{(a')} \in \Lambda^{a'}\big(E^{(a')}\big)$, $f_{l'}^{(a')} \in \Lambda^{a'}\big(F_{l'}^{(a')}\big)$ and $g_{l'}^{(a')} \in \Lambda^{a'}\big(G_{l'}^{(a')}\big) = \Lambda^{a'}\big(F_{l'-1}^{(a')}\big)$.

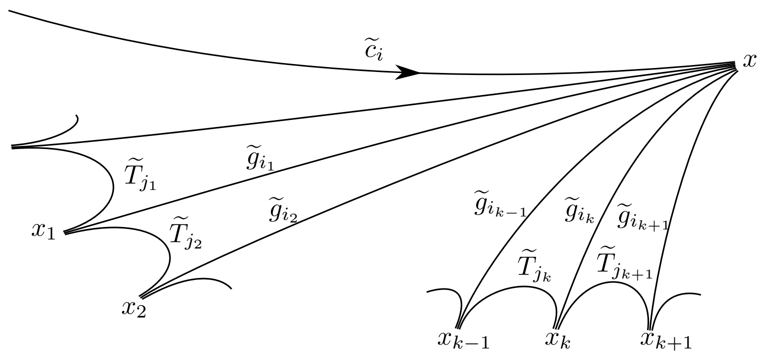


FIGURE 5. Spiraling in the universal cover $\widetilde{S}$

Lemma 14.

$$\overline{\sigma}^\rho_a(g_{i_l}) = \log \left| \frac{e^{(a)} \wedge f_l^{(n-a-1)} \wedge g_l^{(1)}}{e^{(a-1)} \wedge f_l^{(n-a)} \wedge g_l^{(1)}} \; \frac{e^{(a-1)} \wedge f_{l+1}^{(1)} \wedge g_{l+1}^{(n-a)}}{e^{(a)} \wedge f_{l+1}^{(1)} \wedge g_{l+1}^{(n-a-1)}} \; \frac{e^{(a)} \wedge f_l^{(n-a)}}{e^{(a)} \wedge g_{l+1}^{(n-a)}} \; \frac{e^{(a+1)} \wedge g_{l+1}^{(n-a-1)}}{e^{(a+1)} \wedge f_l^{(n-a-1)}} \right|$$

Proof. The right-hand side of the equation is clearly independent of the choices of elements $e^{(a')} \in \Lambda^{a'}\big(E^{(a')}\big)$, $f_{l'}^{(a')} \in \Lambda^{a'}\big(F_{l'}^{(a')}\big)$ and $g_{l'}^{(a')} \in \Lambda^{a'}\big(F_{l'-1}^{(a')}\big)$. It is equal to $\log D_a(E, F_l, F_{l-1}, F_{l+1})$ in the special case where $g_{l'}^{(a')} = f_{l'-1}^{(a')}$, and therefore in all cases.

(The absolute value is introduced so that, here and later in the arguments, we do not have to worry about sign changes when we split ratios or permute terms in the wedge products.)

By definition of $\sigma^\rho_a(g_{i_l})$, this quantity $\log D_a(E, F_l, F_{l-1}, F_{l+1})$ is equal to $\sigma^\rho_a(g_{i_l})$ when the orientation of $\widetilde{g}_{i_l}$ points towards $x \in \partial_\infty \widetilde{S}$, and is equal to $\sigma^\rho_{n-a}(g_{i_l})$ when g_{i_l} is oriented away from x. It is therefore equal to $\overline{\sigma}^\rho_a(g_{i_l})$ in all cases. $\square$

Lemma 14 enables us to split $\overline{\sigma}^\rho_a(g_{i_l})$ according to the respective contributions of the triangles $\widetilde{T}_{i_l}$ and $\widetilde{T}_{i_{l+1}}$:

$$\overline{\sigma}^\rho_a(g_{i_l}) = \log \left| \frac{e^{(a)} \wedge f_l^{(n-a-1)} \wedge g_l^{(1)}}{e^{(a-1)} \wedge f_l^{(n-a)} \wedge g_l^{(1)}} \frac{e^{(a)} \wedge f_l^{(n-a)}}{e^{(a+1)} \wedge f_l^{(n-a-1)}} \right| + \log \left| \frac{e^{(a-1)} \wedge f_{l+1}^{(1)} \wedge g_{l+1}^{(n-a)}}{e^{(a)} \wedge f_{l+1}^{(1)} \wedge g_{l+1}^{(n-a-1)}} \frac{e^{(a+1)} \wedge g_{l+1}^{(n-a-1)}}{e^{(a)} \wedge g_{l+1}^{(n-a)}} \right|$$

Summing over l and grouping terms according to the contribution of each triangle gives

$$\begin{aligned}\sum_{l=1}^{k} \overline{\sigma}_a^\rho(g_{i_l}) = \log \left| \frac{e^{(a)} \wedge f_1^{(n-a-1)} \wedge g_1^{(1)}}{e^{(a-1)} \wedge f_1^{(n-a)} \wedge g_1^{(1)}} \frac{e^{(a)} \wedge f_1^{(n-a)}}{e^{(a+1)} \wedge f_1^{(n-a-1)}} \right| \\ + \sum_{l=2}^{k} \log \left| \frac{e^{(a)} \wedge f_l^{(n-a-1)} \wedge g_l^{(1)}}{e^{(a-1)} \wedge f_l^{(n-a)} \wedge g_l^{(1)}} \frac{e^{(a)} \wedge f_l^{(n-a)}}{e^{(a+1)} \wedge f_l^{(n-a-1)}} \right. \\ \left. \frac{e^{(a-1)} \wedge f_l^{(1)} \wedge g_l^{(n-a)}}{e^{(a)} \wedge f_l^{(1)} \wedge g_l^{(n-a-1)}} \frac{e^{(a+1)} \wedge g_l^{(n-a-1)}}{e^{(a)} \wedge g_l^{(n-a)}} \right| \\ + \log \left| \frac{e^{(a-1)} \wedge f_{k+1}^{(1)} \wedge g_{k+1}^{(n-a)}}{e^{(a)} \wedge g_{k+1}^{(n-a)}} \frac{e^{(a+1)} \wedge g_{k+1}^{(n-a-1)}}{e^{(a)} \wedge f_{k+1}^{(1)} \wedge g_{k+1}^{(n-a-1)}} \right|\end{aligned}$$

Consider the last term. Lift $\rho\big([c_i]\big) \in \mathrm{PSL_n}(\mathbb{R})$ to $\rho\big([c_i]\big)' \in \mathrm{SL_n}(\mathbb{R})$. Remembering that $\widetilde{T}_{j_{k+1}}$ is equal to $c_i(\widetilde{T}_{j_1})$, we can choose $f_{k+1}^{(1)} = \rho\big([c_i]\big)'_*(f_1^{(1)})$ and $g_{k+1}^{(a')} = \rho\big([c_i]\big)'_*(g_1^{(a')})$ for every a', where $\rho\big([c_i]\big)'_*\colon \Lambda^b(\mathbb{R}^n) \to \Lambda^b(\mathbb{R}^n)$ is the map induced by $\rho\big([c_i]\big)'\colon \mathbb{R}^n \to \mathbb{R}^n$. Also, $\rho\big([c_i]\big)'$ respects the flag E.

We now use the fact that the spiraling occurs in the direction of the orientation of c_i. As in §2.2, $\rho\big([c_i]\big)'$ has eigenvalues $m_1^\rho(c_i)$, $m_2^\rho(c_i)$, …, $m_n^\rho(c_i)$ with corresponding 1-dimensional eigenspaces L_1, L_2, …, L_n. Because the flag $E = \mathcal{F}_\rho(x)$ is associated with the positive end point x of a component $\widetilde{c}_i$ of the preimage of c_i, it is the stable flag of $\rho\big([c_i]\big)$ by Part (1) of Proposition 10 and

$$E^{(a)} = \bigoplus_{b=1}^{a} L_b.$$

Therefore

$$\rho\big([c_i]\big)'_*\big(e^{(a)}\big) = \bigg(\prod_{b=1}^{a} m_b^\rho(c_i)\bigg) e^{(a)}$$

for every $e^{(a)} \in \Lambda^a(E^{(a)})$.

Then, using the fact that $\rho\big([c_i]\big)' \in \mathrm{SL_n}(\mathbb{R})$ acts by the identity on $\Lambda^n(\mathbb{R}^n)$ for the second equation,

$$\begin{aligned}&\log \left| \frac{e^{(a-1)} \wedge f_{k+1}^{(1)} \wedge g_{k+1}^{(n-a)}}{e^{(a)} \wedge f_{k+1}^{(1)} \wedge g_{k+1}^{(n-a-1)}} \frac{e^{(a+1)} \wedge g_{k+1}^{(n-a-1)}}{e^{(a)} \wedge g_{k+1}^{(n-a)}} \right| \\ &= \log \left| \frac{e^{(a-1)} \wedge \rho\big([c_i]\big)'_*\big(f_1^{(1)}\big) \wedge \rho\big([c_i]\big)'_*\big(g_1^{(n-a)}\big)}{e^{(a)} \wedge \rho\big([c_i]\big)'_*\big(f_1^{(1)}\big) \wedge \rho\big([c_i]\big)'_*\big(g_1^{(n-a-1)}\big)} \frac{e^{(a+1)} \wedge \rho\big([c_i]\big)'_*\big(g_1^{(n-a-1)}\big)}{e^{(a)} \wedge \rho\big([c_i]\big)'_*\big(g_1^{(n-a)}\big)} \right| \\ &= \log \left| \frac{\rho\big([c_i]\big)'^{-1}_*\big(e^{(a-1)}\big) \wedge f_1^{(1)} \wedge g_1^{(n-a)}}{\rho\big([c_i]\big)'^{-1}_*\big(e^{(a)}\big) \wedge f_1^{(1)} \wedge g_1^{(n-a-1)}} \frac{\rho\big([c_i]\big)'^{-1}_*\big(e^{(a+1)}\big) \wedge g_1^{(n-a-1)}}{\rho\big([c_i]\big)'^{-1}_*\big(e^{(a)}\big) \wedge g_1^{(n-a)}} \right| \\ &= \log \left| \frac{m_a^\rho(c_i)}{m_{a+1}^\rho(c_i)} \frac{e^{(a-1)} \wedge f_1^{(1)} \wedge g_1^{(n-a)}}{e^{(a)} \wedge f_1^{(1)} \wedge g_1^{(n-a-1)}} \frac{e^{(a+1)} \wedge g_1^{(n-a-1)}}{e^{(a)} \wedge g_1^{(n-a)}} \right| \\ &= \ell_a^\rho(c_i) + \log \left| \frac{e^{(a-1)} \wedge f_1^{(1)} \wedge g_1^{(n-a)}}{e^{(a)} \wedge f_1^{(1)} \wedge g_1^{(n-a-1)}} \frac{e^{(a+1)} \wedge g_1^{(n-a-1)}}{e^{(a)} \wedge g_1^{(n-a)}} \right|.\end{aligned}$$

Combining with our earlier computation, this yields

$$\sum_{l=1}^{k} \overline{\sigma}_a^\rho(g_{i_l}) = \ell_a^\rho(c_i) + \sum_{l=1}^{k} \log \left| \frac{e^{(a)} \wedge f_l^{(n-a-1)} \wedge g_l^{(1)}}{e^{(a-1)} \wedge f_l^{(n-a)} \wedge g_l^{(1)}} \frac{e^{(a)} \wedge f_l^{(n-a)}}{e^{(a+1)} \wedge f_l^{(n-a-1)}} \right.$$
$$\left. \frac{e^{(a-1)} \wedge f_l^{(1)} \wedge g_l^{(n-a)}}{e^{(a)} \wedge f_l^{(1)} \wedge g_l^{(n-a-1)}} \frac{e^{(a+1)} \wedge g_l^{(n-a-1)}}{e^{(a)} \wedge g_l^{(n-a)}} \right|$$
$$= \ell_a^\rho(c_i) - \sum_{l=1}^{k} \log Q_a(E, F_l, G_l) = \ell_a^\rho(c_i) - \sum_{l=1}^{k} \sum_{b+c=n-a} \log T_{abc}(E, F_l, G_l)$$
$$= \ell_a^\rho(c_i) - \sum_{l=1}^{k} \sum_{b+c=n-a} \tau_{abc}^\rho(T_{j_l}, v_l)$$

where $Q_a(E, F_l, G_l)$ is the quadruple ratio of §1.3, and where we use Lemma 6 for the third equality.

This concludes the proof of Proposition 13 in the first case, when the side of c_i considered is the right-hand side c_i^{right} and where the g_{i_l} spiral on this side in the direction of the orientation of c_i.

Let us now consider the second case, where we are still considering the right-hand side c_i^{right} but where the spiraling occurs in the direction opposite to the orientation of c_i. The arguments are the same except that

$$E^{(a)} = \bigoplus_{b=n-a+1}^{n} L_b.$$

because the flag $E = \mathcal{F}_\rho(x)$ is now associated with the negative end point x of $\widetilde{c}_i$, and is therefore the unstable flag of $\rho\big([c_i]\big)$. It follows that

$$\rho\big([c_i]\big)'_*\big(e^{(a)}\big) = \bigg(\prod_{b=n-a+1}^{n} m_b^\rho(c_i)\bigg) e^{(a)}$$

for every $e^{(a)} \in \Lambda^a(E^{(a)})$.

This leads to the conclusion that

$$\sum_{l=1}^{k} \overline{\sigma}_a^\rho(g_{i_l}) = -\ell_{n-a}^\rho(c_i) - \sum_{l=1}^{k} \sum_{b+c=n-a} \tau_{abc}^\rho(T_{j_l}, v_l).$$

Replacing a by $n-a$ then gives

$$\ell_a^\rho(c_i) = -\sum_{l=1}^{k} \overline{\sigma}_{n-a}^\rho(g_{i_l}) - \sum_{l=1}^{k} \sum_{b+c=a} \tau_{(n-a)bc}^\rho(T_{j_l}, v_l)$$

as desired.

The remaining two cases of Proposition 13, where the side of c_i considered is the left-hand side c_i^{left} are obtained from these first two by reversing the orientation of c_i and using the fact that this replaces $\ell_a^\rho(c_i)$ by $\ell_{n-a}^\rho(c_i)$. □

Remark 15. In the above proof of Proposition 13, it was convenient to use absolute values everywhere so that we did not have to worry about the signs of the quantities inside logarithms. Another method would have been to take the exponential

of the two sides of each equation. This provides a slightly stronger result. For instance, in the first case considered in the proof, this sequence of equations involving exponentials gives

$$\frac{m_a^\rho(c_i)}{m_{a+1}^\rho(c_i)} = \prod_{l=1}^{k} \exp \overline{\sigma}_a(g_{i_l}) \prod_{l=1}^{k} \prod_{b+c=n-a} \exp \tau_{abc}^\rho(T_{j_l}, v_l)$$

and directly proves that the quotient on the left-hand side of the equation is positive, without having to rely on Lemma 9. Similar positivity conclusions hold in the other three cases of the proof.

We will take advantage of this observation in §4.2.

4. Parametrizing the Hitchin component

4.1. **The space of possible invariants.** Recall that we are given a maximal geodesic lamination λ with finitely many leaves, consisting of closed leaves c_i and of infinite leaves g_j whose ends spiral around the c_i. In addition, each leaf c_i or g_j carries an orientation, and each closed leaf c_i is endowed with an arc k_i transverse to λ, cutting c_i in exactly one point, and meeting no other closed leaf c_j with $j \neq i$.

We have associated to a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$ triangle invariants $\tau_{abc}^\rho(T_j, v_j) \in \mathbb{R}$ and shear invariants $\sigma_a^\rho(g_j)$ and $\sigma_a^\rho(c_i)$. Lemma 11 and Proposition 13 provide relations between these invariants. Let $\mathcal{P}$ be the set of all possible such invariants.

More precisely, let $\mathcal{P}$ be the space of functions τ_{abc} and σ_a such that:

(1) for every triple of integers a, b, $c \geqslant 1$ with $a+b+c=n$, τ_{abc} associates a number $\tau_{abc}(T_j, v_j) \in \mathbb{R}$ to each component T_j of the complement $S-\lambda$ and to each vertex v_j of the ideal triangle T_j;
(2) for each integer $a = 1, 2, \dots, n-1$, σ_a associates a number $\sigma_a(c_i)$ or $\sigma_a(g_j) \in \mathbb{R}$ to each leaf c_i or g_j of λ;
(3) for each triangle T_j in the complement $S-\lambda$, the functions τ_{abc} satisfy the *Rotation Condition* stated below;
(4) for every closed leaf c_i and every index $1 \leqslant a \leqslant n-1$, the functions τ_{abc} and σ_a satisfy the *Closed Leaf Equality* condition stated below;
(5) for every closed leaf c_i and every index $1 \leqslant a \leqslant n-1$, the functions τ_{abc} and σ_a satisfy the *Closed Leaf Inequality* condition stated below.

The Rotation Condition just comes from Lemma 11.

Rotation Condition. If the vertices v_j and v_j' of the triangle component T_j of $S-\lambda$ are such that v_j' immediately follows v_j when going clockwise around the boundary of T_j, then

$$\tau_{abc}(T_j, v_j) = \tau_{bca}(T_j, v_j').$$

The Closed Leaf Equality and Closed Leaf Inequality are directly inspired by the formulas of Proposition 13 computing the ρ–lengths $\ell_a^\rho(c_i)$ in terms of the triangle and shear invariants. Let g_{i_1}, g_{i_2}, …, g_{i_k} be the infinite leaves of λ that spiral on the right-hand side c_i^{right} of c_i. Let T_{j_1}, T_{j_2}, …, T_{j_k} be the components of the complement $S-\lambda$ that spiral on this side c_i^{right} and, for each j_l, let v_l be the vertex of the ideal triangle T_{j_l} towards which the spiraling occurs. Similarly, let $g_{i_1'}$, $g_{i_2'}$, …, $g_{i_{k'}'}$ be the infinite leaves of λ that spiral on the left-hand side c_i^{left}, let $T_{j_1'}$,

$T_{j'_2}, \dots, T_{j'_{k'}}$ be the components of the complement $S-\lambda$ that spiral on c_i^{left}, and let v'_l be the vertex of the ideal triangle $T_{j'_l}$ towards which the spiraling occurs. Following Proposition 13, set $\overline{\sigma}_a(g_{i_l}) = \sigma_a(g_{i_l})$ if the leaf g_{i_l} is oriented towards c_i and $\overline{\sigma}_a(g_{i_l}) = \sigma_{n-a}(g_{i_l})$ otherwise, and define

$$L_a^{\mathrm{right}}(c_i) = \sum_{l=1}^{k} \overline{\sigma}_a(g_{i_l}) + \sum_{l=1}^{k} \sum_{b+c=n-a} \tau_{abc}(T_{j_l}, v_l)$$

if the spiraling occurs in the direction of the orientation of c_i, and

$$L_a^{\mathrm{right}}(c_i) = -\sum_{l=1}^{k} \overline{\sigma}_{n-a}(g_{i_l}) - \sum_{l=1}^{k} \sum_{b+c=a} \tau_{(n-a)bc}(T_{j_l}, v_l)$$

if the spiraling occurs in the direction opposite to the orientation of c_i.

Similarly define

$$L_a^{\mathrm{left}}(c_i) = -\sum_{l=1}^{k'} \overline{\sigma}_a(g_{i'_l}) - \sum_{l=1}^{k'} \sum_{b+c=n-a} \tau_{abc}(T_{j'_l}, v'_l)$$

if the spiraling occurs in the direction of the orientation of c_i, and

$$L_a^{\mathrm{left}}(c_i) = \sum_{l=1}^{k'} \overline{\sigma}_{n-a}(g_{i'_l}) + \sum_{l=1}^{k'} \sum_{b+c=a} \tau_{(n-a)bc}(T_{j'_l}, v'_l)$$

otherwise.

The numbers $L_a^{\mathrm{right}}(c_i)$ and $L_a^{\mathrm{right}}(c_i)$ are completely determined by the functions $\tau_{a'b'c'}$ and $\sigma_{a'}$. Proposition 13 says that, when $\tau_{a'b'c'} = \tau^\rho_{a'b'c'}$ and $\sigma_{a'} = \sigma^\rho_{a'}$ are associated to a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$ as in §2, then

$$L_a^{\mathrm{right}}(c_i) = L_a^{\mathrm{left}}(c_i) = \ell_a^\rho(c_i) > 0.$$

This motivates the following conditions.

Closed Leaf Equality. With the above notation,

$$L_a^{\mathrm{right}}(c_i) = L_a^{\mathrm{left}}(c_i)$$

for every closed leaf c_i of λ, and for every index $1 \leqslant a \leqslant n-1$.

Closed Leaf Inequality. With the above notation,

$$L_a^{\mathrm{right}}(c_i) > 0$$

for every closed leaf c_i of λ, and for every index $1 \leqslant a \leqslant n-1$.

The geodesic lamination λ has s closed leaves and t infinite leaves, and its complement $S-\lambda$ consists of u triangles. Also, there are $\frac{(n-1)(n-2)}{2}$ triples of integers a, b, $c \geqslant 1$ with $a+b+c=n$. Therefore, the functions τ_{abc} and σ_a satisfying Conditions (1) and (2) form a vector space of dimension

$$N = 3u\tfrac{(n-1)(n-2)}{2} + (s+t)(n-1).$$

Proposition 16. *The set $\mathcal{P}$ of functions τ_{abc} and σ_a satisfying the above conditions (1–5) is the interior of a convex polytope $\overline{\mathcal{P}} \subset \mathbb{R}^N$.*

Proof. As a subset of $\mathbb{R}^N$, $\mathcal{P}$ is defined by a finite collection of linear equalities and strict inequalities. $\square$

Note that the dimension d of $\overline{\mathcal{P}}$ is strictly less than N. When we refer to the interior of this polytope, we of course mean the topological interior of $\overline{\mathcal{P}}$ in the d–dimensional linear subspace that contains it.

We can formally estimate this dimension d. The Rotation Condition enables us to avoid the reference to vertices in the functions τ_{abc}. The space of functions τ_{abc} satisfying the Rotation Condition is therefore $u\frac{(n-1)(n-2)}{2} = 2(g-1)(n-1)(n-2)$, as $u = 4(g-1)$. As before, the space of functions σ_a has dimension $(s+t)(n-1)$. Since there are $s(n-1)$ Closed Leaf Equalities, the expected dimension of $\mathcal{P}$ should therefore be

$$2(g-1)(n-1)(n-2) + (s+t)(n-1) - s(n-1) = 2(g-1)(n^2-1)$$

if we remember that $t = 6(g-1)$. This formal computation can be justified by an explicit argument showing the independence of the relations. However, the combinatorics involved are somewhat complicated. We will be content with the observation that this dimension $2(g-1)(n^2-1)$ is also the dimension of the Hitchin component $\mathrm{Hit_n}(S)$, and prove that $\mathcal{P}$ is homeomorphic to $\mathrm{Hit_n}(S)$.

In §2, we associated with each Hitchin representation $\rho \in \mathrm{Hit_n}(S)$ functions τ^ρ_{abc} and α^ρ_a as above, and showed that these functions satisfy the Rotation Condition, the Closed Leaf Equalities and the Closed Leaf Inequalities (in §2.6 and §3). In other words, we constructed a map $\Phi\colon \mathrm{Hit_n}(S) \to \mathcal{P}$.

Theorem 17. *The above map*

$$\Phi\colon \mathrm{Hit_n}(S) \to \mathcal{P}$$

is a homeomorphism.

The proof of Theorem 17 will occupy all of the next section §4.2.

4.2. **Proof that the map $\Phi\colon \mathrm{Hit_n}(S) \to \mathcal{P}$ is a homeomorphism.** We begin with a small step.

Lemma 18. *The map $\Phi\colon \mathrm{Hit_n}(S) \to \mathcal{P}$ is continuous.*

Proof. This is an immediate consequence of the Anosov property, which implies that the flag curve $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ depends continuously on the Hitchin representation ρ. See [La, §2]. The reader can also consult [Gui, §2.6], and in particular [GuiW, §5.3]. □

We will construct a right inverse $\mathcal{P} \to \mathrm{Hit_n}(S)$ for Φ. For this, suppose that we are given functions τ_{abc} and σ_a that satisfy the Rotation Condition, the Closed Leaf Equalities and the Closed Leaf Inequalities, namely that define a point of the polytope $\mathcal{P}$. We will construct a Hitchin representation $\rho \in \mathrm{Hit_n}(S)$ whose invariants are exactly these functions τ_{abc} and σ_a.

Our strategy will be to reconstruct the flag curve $\mathcal{F}_\rho$ of §2.2. However, because we do not yet have a Hitchin representation, we will use a weaker version of this flag curve.

Let $\partial_\infty \widetilde{\lambda}$ be the subset of the circle of infinity $\partial_\infty \widetilde{S}$ that consists of the end points of the leaves of the preimage $\widetilde{\lambda} \subset \widetilde{S}$ of the maximal geodesic lamination λ. More generally, if $\widetilde{\lambda}' \subset \widetilde{\lambda}$ is a family of leaves of $\widetilde{\lambda}$, then $\partial_\infty \widetilde{\lambda}' \subset \partial_\infty \widetilde{\lambda}$ consists of the end points of leaves of $\widetilde{\lambda}'$. A *flag decoration* for $\widetilde{\lambda}'$ is a (not necessarily continuous) map $\mathcal{F}\colon \partial_\infty \widetilde{\lambda}' \to \mathrm{Flag}(\mathbb{R}^n)$.

A fundamental example of such a flag decoration of course comes from the restriction to $\partial_\infty \widetilde{\lambda}$ of the flag curve $\mathcal{F}_\rho\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ of a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$. Note that the definition of the triangle invariants $\tau^\rho_{abc}(T, v)$ and the shear invariants $\sigma^\rho_a(g_j)$ and $\sigma^\rho_a(c_i)$ only uses the flag decoration $\mathcal{F}\colon \partial_\infty \widetilde{\lambda} \to \mathrm{Flag}(\mathbb{R}^n)$, and not the full flag curve $\mathcal{F}_\rho$.

We can copy these constructions for a general flag decoration $\mathcal{F}\colon \partial_\infty \widetilde{\lambda}' \to \mathrm{Flag}(\mathbb{R}^n)$. For instance, if $\widetilde{T}_j$ is a triangle component of $\widetilde{S} - \widetilde{\lambda}$ whose three sides are in the sublamination $\widetilde{\lambda}'$, and if $\widetilde{v}_j$ is a vertex of $\widetilde{T}_j$, we can often define triangle invariants

$$\tau^{\mathcal{F}}_{abc}(\widetilde{T}_j, \widetilde{v}_j) = \log T_{abc}\big(\mathcal{F}(\widetilde{v}_j), \mathcal{F}(\widetilde{v}'_j), \mathcal{F}(\widetilde{v}''_j)\big)$$

as in §2.3, where the vertices $\widetilde{v}_j$, $\widetilde{v}'_j$, $\widetilde{v}''_j$ of $\widetilde{T}$ are indexed in clockwise order. Of course, this triangle invariant only makes sense under the assumption that the triple ratio $T_{abc}\big(\mathcal{F}(\widetilde{v}_j), \mathcal{F}(\widetilde{v}'_j), \mathcal{F}(\widetilde{v}''_j)\big)$ is positive.

Similarly, suppose that we are given an isolated leaf $\widetilde{g}_i$ of $\widetilde{\lambda}$ such that the two triangle components of $\widetilde{S} - \widetilde{\lambda}$ that are adjacent to $\widetilde{\lambda}$ have all their sides contained in $\widetilde{\lambda}'$. We can then define

$$\sigma^{\mathcal{F}}_a(\widetilde{g}_i) = \log D_a(E, F, G, G') = \log D_a\big(\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z), \mathcal{F}(z')\big)$$

with the conventions of §2.4. Again, this requires the double product $D_a(E, F, G, G')$ to be positive.

Finally, let $\widetilde{k}_i$ be an arc lifting one of the arcs k_i that are part of the topological data of §2.1. We then define

$$\sigma^{\mathcal{F}}_a(\widetilde{k}_i) = \log D_a(E, F, G, G') = \log D_a\big(\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z), \mathcal{F}(z')\big)$$

with the conventions of §2.5, when the points x, y, z, z' are in $\partial_\infty \widetilde{\lambda}'$ and when the double product $D_a(E, F, G, G')$ is positive.

Note that, without any equivariance condition for the flag decoration $\mathcal{F}$, there is no reason for the invariants $\tau^{\mathcal{F}}_{abc}(\widetilde{T}_j, \widetilde{v}_j)$, $\sigma^{\mathcal{F}}_a(\widetilde{g}_i)$ and $\sigma^{\mathcal{F}}_a(\widetilde{k}_i)$ to be invariant under the action of $\pi_1(S)$ on $\widetilde{S}$.

After these preliminaries on flag decorations, we return to our construction of an inverse for the map $\Phi\colon \mathrm{Hit_n}(S) \to \mathcal{P}$. Consider functions τ_{abc} and σ_a that define a point of the polytope $\mathcal{P}$, namely that satisfy the Rotation Conditions, the Closed Leaf Equalities, and the Closed Leaf Inequalities.

These functions lift to the universal covering $\widetilde{S}$, and define numbers $\tau_{abc}(\widetilde{T}_j, \widetilde{v}_j)$, $\sigma_a(\widetilde{g}_i)$ and $\sigma_a(\widetilde{k}_i) \in \mathbb{R}$ for every triangle component $\widetilde{T}_j$ of $\widetilde{S} - \widetilde{\lambda}$, every vertex $\widetilde{v}_j$ of $\widetilde{T}_j$, every isolated leaf $\widetilde{g}_i$ of $\widetilde{\lambda}$, and every arc $\widetilde{k}_i$ lifting one of the transverse arcs k_i.

Lemma 19. *For every component $\widetilde{T}_j$ of $\widetilde{S} - \widetilde{\lambda}$, there exists a flag decoration $\mathcal{F}\colon \partial_\infty \widetilde{T}_j \to \mathrm{Flag}(\mathbb{R}^n)$ for the boundary of $\widetilde{T}_j$ such that*

$$\tau^{\mathcal{F}}_{abc}(\widetilde{T}_j, \widetilde{v}_j) = \tau_{abc}(\widetilde{T}_j, \widetilde{v}_j)$$

for every vertex $\widetilde{v}_j$ of $\widetilde{T}_j$ and every integers a, b, $c \geqslant 1$ such that $a+b+c = n$ (and where $\partial_\infty \widetilde{T}_j$ consists of the three vertices of $\widetilde{T}_j$).

In addition, $\mathcal{F}$ is unique up to composition with a map $\mathrm{Flag}(\mathbb{R}^n) \to \mathrm{Flag}(\mathbb{R}^n)$ induced by an element of $\mathrm{PGL_n}(\mathbb{R})$.

Proof. For a single vertex $\widetilde{v}_j$, this is just another way of saying that there exists a flag triple whose triple ratios are $\exp \tau_{abc}(\widetilde{T}_j, \widetilde{v}_j)$, as asserted by Proposition 4. The

property for all three vertices of $\widetilde{T}_j$ then follows from the Rotation Condition. The uniqueness property is also a consequence of Proposition 4. □

We now put two adjacent triangles together.

Lemma 20. *Let $\widetilde{T}_j$ and $\widetilde{T}'_{j'}$ be two adjacent components of $\widetilde{S}-\widetilde{\lambda}$, separated by a leaf $\widetilde{g}_i$ of $\widetilde{\lambda}$. Then, if $\widetilde{\lambda}'$ denotes the union of the sides of $\widetilde{T}_j$ and $\widetilde{T}'_{j'}$, there exists a flag decoration $\mathcal{F}\colon \partial_\infty\widetilde{\lambda}' \to \mathrm{Flag}(\mathbb{R}^n)$ such that*

1. *$\tau^{\mathcal{F}}_{abc}(\widetilde{T}_j,\widetilde{v}) = \tau_{abc}(\widetilde{T}_j,\widetilde{v})$ for every vertex $\widetilde{v}$ of $\widetilde{T}_j$ and every integers a, b, $c \geqslant 1$ such that $a+b+c=n$;*
2. *$\tau^{\mathcal{F}}_{abc}(\widetilde{T}'_{j'},\widetilde{v}') = \tau_{abc}(\widetilde{T}'_{j'},\widetilde{v}')$ for every vertex $\widetilde{v}'$ of $\widetilde{T}'_{j'}$ and every integers a, b, $c \geqslant 1$ such that $a+b+c=n$;*
3. *$\sigma^{\mathcal{F}}_a(\widetilde{g}_i) = \sigma_a(\widetilde{g}_i)$ for every integer a with $1 \leqslant a \leqslant n-1$.*

In addition, the flag decoration $\mathcal{F}\colon \partial_\infty\widetilde{\lambda}' \to \mathrm{Flag}(\mathbb{R}^n)$ is unique up to post-composition with the action of an element of $\mathrm{PGL_n}(\mathbb{R})$ on $\mathrm{Flag}(\mathbb{R}^n)$.

Proof. Let $\mathcal{F}\colon \partial_\infty\widetilde{T}_j \to \mathrm{Flag}(\mathbb{R}^n)$ and $\mathcal{F}'\colon \partial_\infty\widetilde{T}'_{j'} \to \mathrm{Flag}(\mathbb{R}^n)$ be given by Lemma 19. We only need to arrange that $\mathcal{F}$ and $\mathcal{F}'$ coincide on the end points of $\widetilde{g}_i$, and that Condition (3) is satisfied.

Let x and y be the positive and negative end points of $\widetilde{g}_i$ and, without loss of generality, assume that $\widetilde{T}_j$ is on the left of $\widetilde{g}_i$ for the orientation of this leaf. Consider the flags $E=\mathcal{F}(x)$, $F=\mathcal{F}(y)$, $E'=\mathcal{F}'(x)$, $F'=\mathcal{F}'(y)\in\mathrm{Flag}(\mathbb{R}^n)$.

Because the triangle invariants $\tau^{\mathcal{F}}_{abc}(\widetilde{T}_{j'},\widetilde{v})$ are defined, the flag triple (E,F,G) associated by $\mathcal{F}$ to the three vertices of $\widetilde{T}_j$ is positive. In particular, the flag pair (E,F) is generic. Similarly, the flag pair (E',F') is also generic.

Therefore, by elementary linear algebra, there exists an element $A\in\mathrm{PGL_n}(\mathbb{R})$ that sends E' to E and F' to F. As a first approximation, we can then define the flag decoration $\mathcal{F}\colon \partial_\infty\widetilde{\lambda}' \to \mathrm{Flag}(\mathbb{R}^n)$ to coincide with the original $\mathcal{F}$ on $\partial_\infty\widetilde{T}_j$, and with $A\circ\mathcal{F}'$ on $\partial_\infty\widetilde{T}'_{j'}$.

If z and z' are the third vertices of $\widetilde{T}_j$ and $\widetilde{T}'_{j'}$, respectively, and if we consider the flags $G=\mathcal{F}(z)$ and $G'=\mathcal{F}(z')=A\circ\mathcal{F}'(z')$, then

$$\sigma^{\mathcal{F}}_a(\widetilde{g}_i) = \log D_a(E,F,G,G').$$

Of course, at this point, there is no guarantee that the double ratio $D_a(E,F,G,G')$ is positive, so that $\sigma^{\mathcal{F}}_a(\widetilde{g}_i)$ does not necessarily make sense.

We first compute the double ratio $D_a(E,F,G,G')$ more explicitly. Choose a basis $e_1, e_2, \dots, e_n$ for $\mathbb{R}^n$ such that each e_a generates the line $E^{(a)}\cap F^{(n-a+1)}$. Express generators for the lines $G^{(1)}$ and $G'^{(1)}$ as $g_1=\sum_{a=1}^n\gamma_a e_a$ and $g'_1=\sum_{a=1}^n\gamma'_a e_a$, respectively. Then, if we use the non-zero elements $e^{(b)}=e_1\wedge e_2\wedge\dots\wedge e_b\in\Lambda^b(E^{(b)})$ and $f^{(b)}=e_{n-b+1}\wedge e_{n-b+2}\wedge\dots\wedge e_n\in\Lambda^b(F^{(b)})$ in our computation,

$$\begin{aligned} D_a(E,F,G,G') &= -\frac{e^{(a)}\wedge f^{(n-a-1)}\wedge g_1}{e^{(a)}\wedge f^{(n-a-1)}\wedge g'_1}\,\frac{e^{(a-1)}\wedge f^{(n-a)}\wedge g'_1}{e^{(a-1)}\wedge f^{(n-a)}\wedge g_1} \\ &= -\frac{\gamma_{a+1}}{\gamma'_{a+1}}\frac{\gamma'_a}{\gamma_a}. \end{aligned}$$

We will now take advantage of the fact that there were many possible choices for $A\in\mathrm{PGL_n}(\mathbb{R})$. Indeed, we can always post-compose A with an element $B\in\mathrm{PGL_n}(\mathbb{R})$ respecting the generic flag pair (E,F). Such a $B\in\mathrm{PGL_n}(\mathbb{R})$ respects

each of the lines $E^{(a)} \cap F^{(n-a+1)}$, and consequently can be represented by a matrix of $\mathrm{GL_n}(\mathbb{R})$ which acts on each L_a by multiplication by $\beta_a \in \mathbb{R}^*$. Replacing A by $B \circ A$ then replaces each γ'_b by $\beta_b \gamma'_b$. We can therefore adjust the matrix $A \in \mathrm{PGL_n}(\mathbb{R})$ so that

$$D_a(E, F, G, G') = \exp \sigma_a(\widetilde{g}_i)$$

for every integer a with $1 \leqslant a \leqslant n-1$, and therefore so that $\sigma_a^{\mathcal{F}}(\widetilde{g}_i) = \sigma_a(\widetilde{g}_i)$ for all a.

For such a choice of A, the flag decoration $\mathcal{F}\colon \partial_\infty \widetilde{\lambda}' \to \mathrm{Flag}(\mathbb{R}^n)$ that coincides with the original $\mathcal{F}$ on $\partial_\infty \widetilde{T}_j$ and with $A \circ \mathcal{F}'$ on $\partial_\infty \widetilde{T}'_{j'}$ then satisfies the conclusion of Lemma 20.

In fact, in the above argument, the adjustment factor B is unique as an element of $\mathrm{PGL_n}(\mathbb{R})$. Using the uniqueness in Lemma 19, the uniqueness part of Lemma 20 easily follows. □

We now put infinitely many adjacent triangles together. Let $\lambda^{\mathrm{closed}}$ denote the union of the closed leaves of λ, and let $\widetilde{\lambda}^{\mathrm{closed}}$ be its preimage in the universal covering $\widetilde{S}$.

Lemma 21. *Let $\widetilde{U}$ be a component of $\widetilde{S} - \widetilde{\lambda}^{\mathrm{closed}}$. Then, there exists a flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde{\lambda} \cap \widetilde{U}) \to \mathrm{Flag}(\mathbb{R}^n)$ such that*

(1) $\tau^{\mathcal{F}}_{abc}(\widetilde{T}_j, \widetilde{v}_j) = \tau_{abc}(\widetilde{T}_j, \widetilde{v}_j)$ *for every component $\widetilde{T}_j$ of $\widetilde{U} - \widetilde{\lambda}$, for every vertex $\widetilde{v}$ of $\widetilde{T}_j$, and for every integers a, b, $c \geqslant 1$ such that $a+b+c=n$;*

(2) $\sigma^{\mathcal{F}}_a(\widetilde{g}_i) = \sigma_a(\widetilde{g}_i)$ *for every leaf $\widetilde{g}_i$ of $\widetilde{\lambda} \cap \widetilde{U}$ and for every integer a with $1 \leqslant a \leqslant n-1$.*

In addition, the flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde{\lambda} \cap \widetilde{U}) \to \mathrm{Flag}(\mathbb{R}^n)$ is unique up to post-composition with the action of an element of $\mathrm{PGL_n}(\mathbb{R})$ on $\mathrm{Flag}(\mathbb{R}^n)$.

Proof. By construction, all the leaves of $\widetilde{\lambda} \cap \widetilde{U}$ are isolated. Looking at the dual tree of the cell decomposition of $\widetilde{U}$ induced by $\widetilde{\lambda} \cap \widetilde{U}$, we can therefore list all the components of $\widetilde{U} - \widetilde{\lambda}$ as $\widetilde{T}_{j_1}, \widetilde{T}_{j_2}, \ldots, \widetilde{T}_{j_k}, \ldots$ in such a way that each $\widetilde{T}_{j_k}$ is adjacent to exactly one $\widetilde{T}_{j_l}$ with $l < k$.

We construct $\mathcal{F}$ on the closure $\widetilde{U}_k$ of $\widetilde{T}_{j_1} \cup \widetilde{T}_{j_2} \cup \cdots \cup \widetilde{T}_{j_k}$, by induction on k. The induction starts with Lemma 19.

Suppose that we have constructed a flag decoration $\mathcal{F}_{k-1}\colon \partial_\infty(\widetilde{\lambda} \cap \widetilde{U}_{k-1}) \to \mathrm{Flag}(\mathbb{R}^n)$ with the desired triangle and shear invariants. By hypothesis, the triangle $\widetilde{T}_{j_k}$ is adjacent to a triangle $\widetilde{T}_{j_l}$ with $l < k$; namely the closures of $\widetilde{T}_{j_k}$ and $\widetilde{T}_{j_l}$ meet along a leaf $\widetilde{g}_i$ of $\widetilde{\lambda} \cap \widetilde{U}$.

Apply Lemma 20 to the two triangles $\widetilde{T}_{j_k}$ and $\widetilde{T}_{j_l}$. This provides a flag decoration $\mathcal{F}'\colon \partial_\infty \widetilde{T}_{j_k} \cup \partial_\infty \widetilde{T}_{j_l} \to \mathrm{Flag}(\mathbb{R}^n)$ whose triangle and shear invariants are as requested. Composing $\mathcal{F}'$ with an appropriate element of $\mathrm{PGL_n}(\mathbb{R})$ (using the uniqueness part of Lemma 19), we can arrange that $\mathcal{F}'$ coincides with $\mathcal{F}_{k-1}$ on $\partial_\infty \widetilde{T}_{j_l}$. Since $\partial_\infty(\widetilde{\lambda} \cap \widetilde{U}_k) = \partial_\infty(\widetilde{\lambda} \cap \widetilde{U}_{k-1}) \cup \partial_\infty \widetilde{T}_{j_k}$, we can then define $\mathcal{F}_k\colon \partial_\infty(\widetilde{\lambda} \cap \widetilde{U}_k) \to \mathrm{Flag}(\mathbb{R}^n)$ to coincide with $\mathcal{F}_{k-1}$ on $\partial_\infty(\widetilde{\lambda} \cap \widetilde{U}_{k-1})$ and with $\mathcal{F}'$ on $\partial_\infty \widetilde{T}_{j_k}$. Then, this flag decoration for $\widetilde{\lambda} \cap \widetilde{U}_k$ has the required triangle and shear invariants, and proves the induction step.

This provides a family of flag decorations $\mathcal{F}_k\colon \partial_\infty(\widetilde{\lambda} \cap \widetilde{U}_k) \to \mathrm{Flag}(\mathbb{R}^n)$ such that $\mathcal{F}_k$ coincides with $\mathcal{F}_l$ on $\partial_\infty(\widetilde{\lambda} \cap \widetilde{U}_l)$ whenever $k > l$. Since $\widetilde{U}$ is the union of the $\widetilde{U}_k$,

these $\mathcal{F}_k$ then give a flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde\lambda \cap \widetilde U) \to \mathrm{Flag}(\mathbb{R}^n)$ with the requested triangle and shear invariants.

The uniqueness part of the statement easily follows from that of Lemma 20. □

Lemma 22. *Under the hypotheses and conclusions of Lemma 21, let $\pi_1(U)$ be the stabilizer of $\widetilde U$ in $\pi_1(S)$ (corresponding to the fundamental group of the projection U of $\widetilde U$ onto S, for appropriate base points). Then the flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde\lambda \cap \widetilde U) \to \mathrm{Flag}(\mathbb{R}^n)$ is ρ–equivariant for a unique homomorphism $\rho\colon \pi_1(U) \to \mathrm{PGL_n}(\mathbb{R})$.*

Proof. Consider an element $\gamma \in \pi_1(U)$.

Let $\widetilde T_j$ be one of the components of $\widetilde U - \widetilde\lambda$, projecting onto a component T_j of $S - \lambda$. By construction, for every vectex $\widetilde v_j$ of $\widetilde T_j$,

$$\tau^{\mathcal{F}}_{abc}(\gamma\widetilde T_j, \gamma\widetilde v_j) = \tau_{abc}(\gamma\widetilde T_j, \gamma\widetilde v_j) = \tau_{abc}(T_j, v_j) = \tau_{abc}(\widetilde T_j, \widetilde v_j) = \tau^{\mathcal{F}}_{abc}(\widetilde T_j, \widetilde v_j)$$

where v_j is the vertex of T_j corresponding to $\widetilde v_j$. By the uniqueness part of Lemma 19, there consequently exists an element $\rho(\gamma) \in \mathrm{PGL_n}(\mathbb{R})$ sending the positive flag triple associated with $(\gamma\widetilde T_j, \gamma\widetilde v_j)$ by the flag decoration $\mathcal{F}$.

Using the Rotation Condition, this element $\rho(\gamma)$ does not depend on the choice of the vertex $\widetilde v_j$. Also, reconstructing $\widetilde U$ one triangle component at a time as in the proof of Lemma 21, and applying each time the uniqueness property of Lemma 20, we see that $\rho(\gamma)$ is also independent of the triangle $\widetilde T_j$.

Since every point of $\partial_\infty(\widetilde\lambda \cap \widetilde U)$ is a vertex of some triangle component of $\widetilde S - \widetilde\lambda$, it follows that $\mathcal{F}(\gamma x) = \rho(\gamma)\mathcal{F}(x)$ for every $x \in \partial_\infty(\widetilde\lambda \cap \widetilde U)$.

This defines a map $\rho\colon \pi_1(U) \to \mathrm{PGL_n}(\mathbb{R}))$, which is easily seen to be a group homomorphism. The above property shows that the flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde\lambda \cap \widetilde U) \to \mathrm{Flag}(\mathbb{R}^n)$ is ρ–equivariant.

The uniqueness of ρ is an immediate consequence of the fact that the action of $\mathrm{PGL_n}(\mathbb{R})$ on the generic flag triples is free. □

So far, the arguments were essentially those of Fock and Goncharov in [FoG]. The next step involves a few new twists, and uses the Closed Leaf Equalities and Inequalities in a critical way.

Lemma 23. *Let $\widetilde U_1$ and $\widetilde U_2$ be two adjacent components of $\widetilde S - \widetilde\lambda^{\mathrm{closed}}$, whose closures meet along a component $\widetilde c_i$ of $\widetilde\lambda^{\mathrm{closed}}$. Then, if $\widetilde V$ denotes the union of $\widetilde U_1$, $\widetilde U_2$ and $\widetilde c_i$, there exists a flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde\lambda \cap \widetilde V) \to \mathrm{Flag}(\mathbb{R}^n)$ such that*

1. *$\tau^{\mathcal{F}}_{abc}(\widetilde T_j, \widetilde v_j) = \tau_{abc}(\widetilde T_j, \widetilde v_j)$ for every component $\widetilde T_j$ of $\widetilde V - \widetilde\lambda$, for every vertex $\widetilde v$ of $\widetilde T_j$, and for every integers a, b, $c \geqslant 1$ such that $a + b + c = n$;*
2. *$\sigma^{\mathcal{F}}_a(\widetilde g_i) = \sigma_a(\widetilde g_i)$ for every leaf $\widetilde g_i$ of $\widetilde\lambda \cap \widetilde U_1$ or $\widetilde\lambda \cap \widetilde U_2$, and for every integer a with $1 \leqslant a \leqslant n-1$;*
3. *if c_i is the closed leaf of λ that is the image of $\widetilde c_i$, if k_i is the transverse arc cutting c_i in one point that is part of the topological data, and if the arc $\widetilde k_i$ lifts k_i to $\widetilde S$ and meets $\widetilde c_i$ in one point, then $\sigma^{\mathcal{F}}_a(\widetilde k_i) = \sigma_a(\widetilde k_i)$ for every integer a with $1 \leqslant a \leqslant n-1$.*

In addition, the flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde\lambda \cap \widetilde V) \to \mathrm{Flag}(\mathbb{R}^n)$ is unique up to post-composition with the action of an element of $\mathrm{PGL_n}(\mathbb{R})$ on $\mathrm{Flag}(\mathbb{R}^n)$.

Note that Condition (3) has to be satisfied for every arc $\widetilde{k}_i$ lifting k_i as indicated. This condition is much stronger than one could think at first glance. As we will see, it requires that the Closed Leaf Equalities hold for the functions τ_{abc} and σ_a.

Proof. Let $\mathcal{F}_1 : \partial_\infty(\widetilde{\lambda}\cap\widetilde{U}_1) \to \mathrm{Flag}(\mathbb{R}^n)$ and $\mathcal{F}_2 : \partial_\infty(\widetilde{\lambda}\cap\widetilde{U}_2) \to \mathrm{Flag}(\mathbb{R}^n)$ be the flag decorations provided by Lemma 21, respectively equivariant with respects to homomorphisms $\rho_1 : \pi_1(U_1) \to \mathrm{PGL_n}(\mathbb{R})$ and $\rho_1 : \pi_1(U_2) \to \mathrm{PGL_n}(\mathbb{R})$ as in Lemma 22.

The leaf $\widetilde{c}_i$ has an infinite cyclic stabilizer in $\pi_1(S)$, generated by the element $[c_i] \in \pi_1(U_1) \cap \pi_1(U_2)$ defined by the choice of an appropriate path connecting the oriented closed curve c_i to the base point used in the definition of $\pi_1(S)$.

The computations of §3 determine the eigenvalues of $\rho_1\big([c_i]\big)$. More precisely, the leaves of $\widetilde{\lambda}\cap\widetilde{U}_1$ that are asymptotic to $\widetilde{c}_i$ all have one endpoint $u \in \partial_\infty(\widetilde{\lambda}\cap\widetilde{U}_1)$ in common, corresponding to the positive or negative end point of $\widetilde{c}_i$ according to the direction of the spiraling. By construction, $\rho_1(c_i) \in \mathrm{PSL_n}(\mathbb{R})$ respects the flag $H_1 = \mathcal{F}_1(u) \in \mathrm{Flag}(\mathbb{R}^n)$ associated to u by the flag decoration $\mathcal{F}_1$. Lift $\rho_1\big([c_i]\big) \in \mathrm{PGL_n}(\mathbb{R})$ to $\rho_1\big([c_i]\big)' \in \mathrm{GL_n}(\mathbb{R})$. We then consider the eigenvalue $m_a^{\rho_1}(c_i)$ of $\rho_1\big([c_i]\big)'$ such that $\rho_1\big([c_i]\big)'$ acts by multiplication by $m_a^{\rho_1}(c_i)$ on $H_1^{(a)}/H_1^{(a+1)} \cong \mathbb{R}$ if u is the positive end point of $\widetilde{c}_i$, and on $H_1^{(n-a+1)}/H_1^{(n-a)} \cong \mathbb{R}$ if u is the negative end point of $\widetilde{c}_i$.

The formulas of §3 then compute each ratio $\frac{m_a^{\rho_1}(c_i)}{m_{a+1}^{\rho_1}(c_i)}$ in terms of the triangle invariants of the components of $\widetilde{U}_1 - \widetilde{\lambda}$ and of the shear invariants of the leaves of $\widetilde{\lambda}\cap\widetilde{U}_1$. We cannot quite apply Proposition 13 as is, because we do not (yet) know that ρ_1 is the restriction of a Hitchin representation. However, as observed in Remark 15, the arguments of the proof of this statement straightforwardly apply to the current case as well. The conclusion is then that, if $L_a^{\mathrm{left}}(c_i)$ and $L_a^{\mathrm{right}}(c_i)$ are defined as for the Closed Leaf Equalities and Inequalities in §4.1, the quotient $\frac{m_a^{\rho_1}(c_i)}{m_{a+1}^{\rho_1}(c_i)}$ is equal to $\exp L_a^{\mathrm{left}}(c_i)$ or $\exp L_a^{\mathrm{right}}(c_i)$ according to whether $\widetilde{U}_1$ is to the left or to the right of $\widetilde{c}_i$ for the orientation of c_i.

Since the functions τ_{abc} and σ_a satisfy the Closed Leaf Inequalities, each of these ratios $\frac{m_a^{\rho_1}(c_i)}{m_{a+1}^{\rho_1}(c_i)} = \exp L_a^{\mathrm{left/right}}(c_i)$ is strictly greater than 1. In particular, the eigenvalues $m_a^{\rho_1}(c_i)$ are all distinct, and $\rho_1(c_i)'$ is diagonalizable. Also, the eigenspace L_a corresponding to the eigenvalue $m_a^{\rho_1}(c_i)$ is 1–dimensional. Consider the flags E_1, $F_1 \in \mathrm{Flag}(\mathbb{R}^n)$ defined by the property that $E_1^{(a)} = \sum_{b=1}^a L_a$ and $F_1^{(a)} = \sum_{b=n-a+1}^n L_a$. Note that our original flag $H_1 = \mathcal{F}_1(u)$ is equal to either E_1 or F_1, according to whether u is the positive or negative end point of $\widetilde{c}_i$.

Switching now to $\widetilde{U}_2$, the same argument provides two flags E_2, $F_2 \in \mathrm{Flag}(\mathbb{R}^n)$ invariant under $\rho_2\big([c_i]\big)$ and eigenvalues $m_a^{\rho_2}(c_i) > 0$ of a lift $\rho_2(c_i)' \in \mathrm{GL_n}(\mathbb{R})$ of $\rho_2\big([c_i]\big) \in \mathrm{PGL_n}(\mathbb{R})$ such that $\rho_2(c_i)'$ acts by multiplication of $m_a^{\rho_2}(c_i)$ on $E_2^{(a)}/E_2^{(a+1)} \cong F_2^{(n-a+1)}/F_2^{(n-a)} \cong \mathbb{R}$.

We now use the fact that the functions τ_{abc} and σ_a satisfy the Closed Leaf Equalities associated with the closed leaf c_i. This implies that $\frac{m_a^{\rho_1}(c_i)}{m_{a+1}^{\rho_1}(c_i)} = \frac{m_a^{\rho_2}(c_i)}{m_{a+1}^{\rho_2}(c_i)}$ for every a. As a consequence, the lift $\rho_2\big([c_i]\big)' \in \mathrm{GL_n}(\mathbb{R})$ of $\rho_2\big([c_i]\big) \in \mathrm{PGL_n}(\mathbb{R})$ can be chosen so that it has the same eigenvalues as $\rho_1\big([c_i]\big)$, and therefore so that it is conjugate to $\rho_1\big([c_i]\big)$ by a matrix $A \in \mathrm{GL_n}(\mathbb{R})$.

The matrix $A \in \mathrm{GL_n}(\mathbb{R})$ sends the eigenspaces of $\rho_2\big([c_i]\big)$ to the eigenspaces of $\rho_1\big([c_i]\big)$, and the induced map $\mathrm{Flag}(\mathbb{R}^n) \to \mathrm{Flag}(\mathbb{R}^n)$ therefore sends E_2 to E_1 and F_2 to F_1.

The set $\partial_\infty(\widetilde\lambda \cap \widetilde V)$ is the union of $\partial_\infty(\widetilde\lambda \cap \widetilde U_1)$, of $\partial_\infty(\widetilde\lambda \cap \widetilde U_2)$ and of the two end points of $\widetilde c_i$. (In fact, $\partial_\infty(\widetilde\lambda \cap \widetilde U_1)$ already contains one of the end points of $\widetilde c_i$, and so does $\partial_\infty(\widetilde\lambda \cap \widetilde U_2)$.) We can therefore define a flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde\lambda \cap \widetilde V) \to \mathrm{Flag}(\mathbb{R}^n)$ by the property that it coincides with $\mathcal{F}_1$ on $\partial_\infty(\widetilde\lambda \cap \widetilde U_1)$, it coincides with $A \circ \mathcal{F}_2$ on $\partial_\infty(\widetilde\lambda \cap \widetilde U_2)$, and it sends the positive and negative end points of $\widetilde c_i$ to $E_1 = A(E_2)$ and $F_1 = A(F_2) \in \mathrm{Flag}(\mathbb{R}^n)$, respectively.

This flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde\lambda \cap \widetilde V) \to \mathrm{Flag}(\mathbb{R}^n)$ clearly satisfies Conditions (1) and (2) of Lemma 23, since these conditions only involve subsets of $\widetilde U_1$ and $\widetilde U_2$. We now have to worry about Condition (3).

Choose a lift $\widetilde k_i \subset \widetilde S$ of the arc k_i that meets $\widetilde c_i$ in one point. At this point, we still have a certain amount of flexibility in the construction of the flag decoration $\mathcal{F}$, since we can replace A by $B \circ A$, where B commutes with $\rho_1\big([c_i]\big)$ and stabilizes the generic flag pair (E_1, F_1). As in the proof of Lemma 20, we can use this flexibility to guarantee that $\sigma_a^{\mathcal{F}}(\widetilde k_i) = \sigma_a(\widetilde k_i)$. The proof is essentially identical to the one used in the proof of Lemma 20, with only minor differences in the notation, so we will not repeat it here.

This takes care of one of the lifts $\widetilde k_i$ of the arc k_i. If $\widetilde k_i'$ is any other lift of k_i that meets $\widetilde c_i$ in one point, there exists a power $[c_i]^k$ of $[c_i] \in \pi_1(S)$ such that $\widetilde k_i' = [c_i]^k \widetilde k_i$. Note that $[c_i] \in \pi_1(U_1) \cap \pi_1(U_2)$ respects $\widetilde V$ and $\widetilde\lambda \cap \widetilde V$ and, by construction, the flag decoration $\mathcal{F}\colon \partial_\infty(\widetilde\lambda \cap \widetilde V) \to \mathrm{Flag}(\mathbb{R}^n)$ is equivariant with respect to $\rho_1\big([c_i]\big) = A \circ \rho_2\big([c_i]\big) \circ A^{-1} \in \mathrm{PGL_n}(\mathbb{R})$. Using the fact that the original flag decorations $\mathcal{F}_1$ and $\mathcal{F}_2$ are respectively ρ_1– and ρ_2–equivariant, it follows that $\sigma_a^{\mathcal{F}}(\widetilde k_i') = \sigma_a^{\mathcal{F}}(\widetilde k_i) = \sigma_a(\widetilde k_i)$.

As in the proof of Lemma 20, the uniqueness of the flag decoration $\mathcal{F}$ up to the action of $\mathrm{PGL_n}(\mathbb{R})$ follows from the uniqueness of $\mathcal{F}_1$ and $\mathcal{F}_2$, and from the fact that the adjustment factor B is unique in $\mathrm{PGL_n}(\mathbb{R})$. □

We are now ready to construct the full flag decoration $\mathcal{F}\colon \partial_\infty \widetilde\lambda \to \mathrm{Flag}(\mathbb{R}^n)$ that we need.

Lemma 24. *There exists a flag decoration $\mathcal{F}\colon \partial_\infty \widetilde\lambda \to \mathrm{Flag}(\mathbb{R}^n)$ such that*

(1) $\tau_{abc}^{\mathcal{F}}(\widetilde T_j, \widetilde v_j) = \tau_{abc}(\widetilde T_j, \widetilde v_j)$ *for every component $\widetilde T_j$ of $\widetilde S - \widetilde\lambda$, for every vertex $\widetilde v$ of $\widetilde T_j$, and for every integers a, b, $c \geqslant 1$ such that $a+b+c=n$;*

(2) $\sigma_a^{\mathcal{F}}(\widetilde g_i) = \sigma_a(\widetilde g_i)$ *for every isolated leaf $\widetilde g_i$ of $\widetilde\lambda$ and for every integer a with $1 \leqslant a \leqslant n-1$;*

(3) $\sigma_a^{\mathcal{F}}(\widetilde k_i) = \sigma_a(\widetilde k_i)$ *for every arc $\widetilde k_i \subset \widetilde S$ lifting one of the transverse arcs k_i that are part of the topological data, and for every integer a with $1 \leqslant a \leqslant n-1$.*

In addition, $\mathcal{F}$ is unique up to post-composition by the map $\mathrm{Flag}(\mathbb{R}^n) \to \mathrm{Flag}(\mathbb{R}^n)$ induced by an element of $\mathrm{PGL_n}(\mathbb{R})$.

Proof. The argument is very similar to the one used in the proof of Lemma 21. List the components of $\widetilde S - \widetilde\lambda$ as $\widetilde U_1, \widetilde U_2, \ldots, \widetilde U_k, \ldots$ in such a way that each $\widetilde U_k$ is adjacent to exactly one $\widetilde U_l$ with $l < k$. One then construct $\mathcal{F}$ on $\widetilde U_1 \cup \widetilde U_2 \cup \cdots \cup \widetilde U_k$ by induction on k, using Lemma 23 at each stage. □

Lemma 25. *Under the hypotheses and conclusions of Lemma 24, there exists a unique homomorphism* $\rho\colon \pi_1(S) \to \mathrm{PGL_n}(\mathbb{R})$ *for which the flag decoration* $\mathcal{F}\colon \partial_\infty \widetilde{\lambda} \to \mathrm{Flag}(\mathbb{R}^n)$ *is* ρ*–equivariant.*

Proof. The proof is essentially identical to that of Lemma 22, using the uniqueness statements of Lemmas 23 and 24. We omit the details. □

Before passing to the next step, we note the following elementary fact.

Lemma 26. *Given generic flag triples* (E, F, G) *and* (E', F', G')*, there exists a unique element of* $\mathrm{PGL_n}(\mathbb{R})$ *that sends the flag* E *to the flag* E'*, the flag* F *to the flag* F'*, and the line* $G^{(1)}$ *to the line* $G'^{(1)}$*.* □

Lemma 26 will enable us to freeze the "$\mathrm{PGL_n}(\mathbb{R})$–ambiguity" in the construction of the flag decoration $\mathcal{F}\colon \partial_\infty \widetilde{\lambda} \to \mathrm{Flag}(\mathbb{R}^n)$ of Lemma 24, and of the homomorphism $\rho\colon \pi_1(S) \to \mathrm{PGL_n}(\mathbb{R})$ of Lemma 25.

We begin with an arbitrary Hitchin representation $\rho_0\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$ and its associated flag curve $\mathcal{F}_{\rho_0}\colon \partial_\infty \widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$. We also select an arbitrary component $\widetilde{T}_{i_0}$ of $\widetilde{S} - \widetilde{\lambda}$, and let x_0, y_0, $z_0 \in \partial_\infty \widetilde{S}$ be the vertices of this triangle. Finally, we consider the positive flag triple (E_0, F_0, G_0) where $E_0 = \mathcal{F}_{\rho_0}(x_0)$, $F_0 = \mathcal{F}_{\rho_0}(y_0)$ and $G_0 = \mathcal{F}_{\rho_0}(z_0)$.

By Lemma 26, the flag decoration $\mathcal{F}\colon \partial_\infty \widetilde{\lambda} \to \mathrm{Flag}(\mathbb{R}^n)$ can be modified by an element of $\mathrm{PGL_n}(\mathbb{R})$ so that $\mathcal{F}(x_0) = E_0$, $\mathcal{F}(y_0) = F_0$ and $\mathcal{F}(z_0)^{(1)} = G_0^{(1)}$. With such a normalization, the flag decoration $\mathcal{F}$ is now uniquely determined. So is the homomorphism $\rho\colon \pi_1(S) \to \mathrm{PGL_n}(\mathbb{R})$.

Lemma 27. *With the above normalization, the homomorphism* ρ *of Lemma* 25 *is valued in* $\mathrm{PSL_n}(\mathbb{R})$ *and is a Hitchin representation.*

Without the above normalization, we would only conclude that ρ is conjugate to a Hitchin representation by an element of $\mathrm{PGL_n}(\mathbb{R})$. This does not make any difference when n is odd, since $\mathrm{PGL_n}(\mathbb{R}) = \mathrm{PSL_n}(\mathbb{R})$ in this case. However, when n is even, the quotient $\mathrm{PGL_n}(\mathbb{R})/\mathrm{PSL_n}(\mathbb{R}) = \mathbb{Z}_2$ acts by conjugation on the character variety $\mathcal{R}_{\mathrm{PSL_n}(\mathbb{R})}(S)$, and sends the Hitchin component $\mathrm{Hit_n}(S)$ to a different component.

Proof. We will use a continuity and connexity argument.

If we examine the proofs of Lemmas 19–24 that lead to the construction of the flag decoration $\mathcal{F}\colon \partial_\infty \widetilde{\lambda} \to \mathrm{Flag}(\mathbb{R}^n)$, we see that $\mathcal{F}$ depends continuously on the functions τ_{abc} and σ_a. More precisely, if $\mathcal{F}\colon \partial_\infty \widetilde{\lambda} \to \mathrm{Flag}(\mathbb{R}^n)$ is normalized as above, then for every $u \in \partial_\infty \widetilde{S}$ the flag $\mathcal{F}(u) \in \mathrm{Flag}(\mathbb{R}^n)$ depends continuously on the finitely many parameters $\tau_{abc}(T_j, v_j)$, $\sigma_a(g_i)$, $\sigma(c_i) \in \mathbb{R}$.

The normalization of the flag decoration $\mathcal{F}$ provides a normalization of the homomorphism $\rho\colon \pi_1(S) \to \mathrm{PGL_n}(\mathbb{R})$ for which $\mathcal{F}$ is ρ–equivariant. Indeed, if x, y, $z \in \partial_\infty \widetilde{S}$ are the vertices of the base triangle $\widetilde{T}_{i_0}$ that we have chosen and if $\gamma \in \pi_1(S)$, $\rho(\gamma)$ is the unique element of $\mathrm{PGL_n}(\mathbb{R})$ sending $\mathcal{F}(x)$ to $\mathcal{F}(\gamma x)$, $\mathcal{F}(y)$ to $\mathcal{F}(\gamma y)$ and $\mathcal{F}(z)$ to $\mathcal{F}(\gamma z)$ in $\mathrm{Flag}(\mathbb{R}^n)$. In particular, this proves that ρ depends continuously on the functions τ_{abc} and σ_a.

If $\mathcal{P}$ is the polytope of Proposition 16, consisting of all functions τ_{abc} and σ_a satisfying the Rotation Condition, the Closed Leaf Equalities and the Closed Leaf

Inequalities, we consequently have constructed a continuous map

$$\Psi\colon \mathcal{P} \to \{\text{homomorphisms } \rho\colon \pi_1(S) \to \mathrm{PGL_n}(\mathbb{R})\}.$$

Now, let us return to the Hitchin representation $\rho_0\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$ used to normalize the flag decoration $\mathcal{F}$ and the homomorphism ρ. The triangle and shear invariants $\tau_{abc}^{\rho_0}$ and $\sigma_a^{\rho_0}$ of ρ_0 define a point P_0 of the polytope $\mathcal{P}$. Because our choice of normalization is specially taylored for ρ_0, the normalized flag decoration $\mathcal{F}\colon \partial_\infty\widetilde{\lambda} \to \mathrm{Flag}(\mathbb{R}^n)$ associated with $P_0 \in \mathcal{P}$ coincides with the restriction of the flag curve $\mathcal{F}_{\rho_0}$ to $\partial_\infty\widetilde{\lambda}$. As a consequence, $\Psi(P_0) = \rho_0$.

Therefore, there is at least one point $P_0 \in \mathcal{P}$ such that the homomorphism $\rho_0 = \Psi(P_0)$ is valued in $\mathrm{PSL_n}(\mathbb{R})$ (and not just in $\mathrm{PGL_n}(\mathbb{R})$) and is a Hitchin representation. Because the convex polytope $\mathcal{P}$ is connected, we conclude by continuity that for every $P \in \mathcal{P}$ the homomorphism $\Psi(P)$ is valued in $\mathrm{PSL_n}(\mathbb{R})$. Also, the Hitchin representations form a whole component of $\{$homomorphisms $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})\}$. The same continuity and connexity argument then shows that every $\Psi(P)$ is a Hitchin representation. $\square$

Composing Ψ with the quotient map under the action of $\mathrm{PSL_n}(\mathbb{R})$ therefore provides a continuous map

$$\overline{\Psi}\colon \mathcal{P} \to \mathrm{Hit_n}(S)$$

from the convex polytope P to the Hitchin component $\mathrm{Hit_n}(S)$.

Lemma 28. *The map $\overline{\Psi}$ is an inverse of $\Phi\colon \mathrm{Hit_n}(S) \to \mathcal{P}$.*

Proof. Let P be a point of $\mathcal{P}$, consisting of functions τ_{abc} and σ_a. We just proved that $\rho = \Psi(P)$ is a Hitchin representation. Let $\mathcal{F}_\rho\colon \partial_\infty\widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ be its flag curve.

Let $\mathcal{F}\colon \partial_\infty\widetilde{\lambda} \to \mathrm{Flag}(\mathbb{R}^n)$ be the normalized flag decoration used in the construction of ρ. Every point $x \in \partial_\infty\widetilde{\lambda}$ is an end point of a component $\widetilde{c}_i$ of a closed leaf c_i of λ, and this component $\widetilde{c}_i$ is invariant under an element $[c_i] \in \pi_1(S)$ represented by c_i suitably connected to the base point by a path. By construction (see the proof of Lemma 23), $\mathcal{F}(x)$ is the stable or unstable flag of $\rho\big([c_i]\big)$, according to whether x is the positive or negative endpoint of $\widetilde{c}_i$. Consequently, the flag decoration $\mathcal{F}$ is just the restriction of the flag curve $\mathcal{F}_\rho$ to $\partial_\infty\widetilde{\lambda}$.

It follows that the triangle and shear invariants τ_{abc}^ρ and σ_a^ρ of ρ are equal to the triangle and shear invariants of the flag decoration $\mathcal{F}$, namely are equal to the functions τ_{abc} and σ_a we started with. This can be rephrased as $\Phi\big(\overline{\Psi}(P)\big) = P$.

Since this holds for every $P \in \mathcal{P}$, this proves that $\Phi \circ \overline{\Psi} = \mathrm{Id}_{\mathcal{P}}$.

Conversely, let $[\rho] \in \mathrm{Hit_n}(S)$ be represented by a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL_n}(\mathbb{R})$. The image $P = \Phi\big([\rho]\big) \in \mathcal{P}$ is defined by the triangle and shear invariants τ_{abc}^ρ and σ_z^ρ of ρ.

To determine $\rho' = \Psi(P)$, we need a normalized flag decoration whose triangle and shear invariants correspond to P, namely are equal to τ_{abc}^ρ and σ_a^ρ. The restriction to $\partial_\infty\widetilde{\lambda}$ of the flag curve $\mathcal{F}_\rho\colon \partial_\infty\widetilde{S} \to \mathrm{Flag}(\mathbb{R}^n)$ has the correct invariants, but is not normalized. If x_0, y_0, z_0 are the vertices of the component $\widetilde{T}_{i_0}$ of $\widetilde{S} - \widetilde{\lambda}$ that we used as a base triangle in the normalization, Lemma 26 says that there exists a unique $A_\rho \in \mathrm{PGL_n}(\mathbb{R})$ that sends the flag $\mathcal{F}_\rho(x)$ to E_0, the flag $\mathcal{F}_\rho(y)$ to F_0 and the line $\mathcal{F}_\rho(z)^{(1)}$ to $G_0^{(1)}$. When ρ is the Hitchin representation ρ_0 used to define the flag triple $(E_0, F_0.G_0)$ in the normalization, A_{ρ_0} is equal to the identity. Since

A_ρ depends continuously on ρ, it is therefore contained in the component of the identity in $\mathrm{PGL_n}(\mathbb{R})$, namely in $\mathrm{PSL_n}(\mathbb{R})$.

Now, the restriction of $A\circ\mathcal{F}_{\rho_0}$ to $\partial_\infty\widetilde\lambda$ is a normalized decoration whose triangle and shear invariants correspond to $P\in\mathcal{P}$. It is equivariant with respect to the homomorphism ρ' obtained by conjugating ρ with A. Therefore, the Hitchin representation $\Psi(P)$ is equal to the homomorphism ρ', and $\overline\Psi(P)=[\rho']=[\rho]\in\mathrm{Hit_n}(S)$.

This proves that $\overline\Psi\big(\Phi([\rho])\big)=[\rho]$ for every $[\rho]\in\mathrm{Hit_n}(S)$, namely that $\overline\Psi\circ\Phi=\mathrm{Id}_{\mathrm{Hit_n}(S)}$. Since $\overline\Psi\circ\Phi=\mathrm{Id}_{\mathrm{Hit_n}(S)}$ and $\Phi\circ\overline\Psi=\mathrm{Id}_{\mathcal{P}}$, this proves that $\overline\Psi$ is the inverse of Φ. □

This proves that $\Phi\colon\mathrm{Hit_n}(S)\to\mathcal{P}$ is a homeomorphism.

The following proposition completes the proof of Theorem 17.

Proposition 29. *The map $\Phi\colon\mathrm{Hit_n}(S)\to\mathcal{P}$ and its inverse $\overline\Psi$ are real analytic.*

Proof. For a Hitchin representation ρ, the triangle invariant $\tau^\rho_{abc}(T_j,v_j)$ is a (real) analytic function of the three flags $\mathcal{F}_\rho(x)$, $\mathcal{F}_\rho(y)$, $\mathcal{F}_\rho(z)\in\mathrm{Flag}(\mathbb{R}^n)$ associated by the flag curve $\mathcal{F}_\rho$ to the vertices x, y, $z\in\partial_\infty\widetilde S$ of a lift $\widetilde T_i$ of the triangle T_i.

Because the three ends of T_j spiral around closed leaves c_i of the geodesic lamination λ, the vertex $x\in\partial_\infty\widetilde S$ is the stable or unstable fixed point of some element $[c_i]\in\pi_1(S)$ represented by one of the closed leaves c_i. Therefore, by Part (1) of Proposition 10, $\mathcal{F}_\rho(x)$ is the stable or unstable flag of $\rho\big([c_i]\big)\in\mathrm{PSL_n}(\mathbb{R})$. Since the matrix $\rho\big([c_i]\big)$ is an analytic function of ρ, so is its stable or unstable flag $\mathcal{F}_\rho(x)$. The same of course holds for $\mathcal{F}_\rho(y)$ and $\mathcal{F}_\rho(z)$.

This proves that the three flags $\mathcal{F}_\rho(x)$, $\mathcal{F}_\rho(y)$, $\mathcal{F}_\rho(z)$ analytically depend on the representation ρ. It follows that the triangle invariant $\tau^\rho_{abc}(T_j,v_j)$ is an analytic function of ρ.

The same argument shows that each shear invariant $\sigma_a(g_i)$ or $\sigma_a(c_j)$ is also an analytic function of ρ.

This proves that the point $\Phi\big([\rho]\big)\in\mathcal{P}$ represented by the triangle and shear invariants of the Hitchin representation ρ analytically depends on ρ. In other words, $\Phi\colon\mathrm{Hit_n}(S)\to\mathcal{P}$ is analytic.

Conversely, consider our definition of the inverse map $\overline\Psi=\Phi^{-1}\colon\mathcal{P}\to\mathrm{Hit_n}(S)$. Given a point $P\in\mathcal{P}$ represented by functions τ_{abc} and σ_a, we constructed a normalized flag decoration $\mathcal{F}\colon\partial_\infty\widetilde\lambda\to\mathrm{Flag}(\mathbb{R}^n)$. This construction, developed in the proofs of Lemmas 19–24, is very explicit. As a consequence, for every $x\in\partial_\infty\widetilde\lambda$, the flag $\mathcal{F}(x)\in\mathrm{Flag}(\mathbb{R}^n)$ is an analytic function of the point $P\in\mathcal{P}$.

We now consider the Hitchin representation $\rho=\Psi(P)$ with respect to which $\mathcal{F}$ is ρ–equivariant. Let x, y, $z\in\partial_\infty\widetilde\lambda$ be the vertices of a fixed triangle component $\widetilde T_j$ of $\widetilde S-\widetilde\lambda$. Then, for each $\gamma\in\pi_1(S)$, the element $\rho(\gamma)\in\mathrm{PSL_n}(\mathbb{R})$ can be analytically expressed in terms of the six flags $\mathcal{F}_\rho(x)$, $\mathcal{F}_\rho(y)$, $\mathcal{F}_\rho(z)$, $\mathcal{F}_\rho(\gamma x)$, $\mathcal{F}_\rho(\gamma y)$, $\mathcal{F}_\rho(\gamma z)\in\mathrm{Flag}(\mathbb{R}^n)$. As a consequence, $\rho(\gamma)$ analytically depends on the point $P\in\mathcal{P}$. This proves that $\rho=\Psi(P)$ is an analytic function of P.

In other words, the function

$$\Psi\colon\mathcal{P}\to\{\text{homomorphisms }\rho\colon\pi_1(S)\to\mathrm{PGL_n}(\mathbb{R})\}$$

is analytic. Its composition $\overline\Psi$ with the projection to $\mathrm{Hit_n}(S)$ is therefore analytic. □

5. Global relations between triangle invariants

Compared to the classical case of the parametrization of the Teichmüller space $\mathcal{T}(S)$ by shear coordinates, the really new feature in the parametrization of Theorem 17 is provided by the triangle invariants $\tau_{abc}(T_j, v_j)$. A somewhat surprising property of these triangle invariants is that they are not independent of each other, and are constrained by certain linear relations.

Proposition 30. *Let ρ be a Hitchin representation with triangle invariants $\tau^\rho_{abc}(T_j, v_j)$. Then, for every integer a with $1 \leqslant a \leqslant n-1$,*

$$\sum_{j=1}^{u} \sum_{v_j \text{ vertex of } T_j} \left(\sum_{b+c=n-a} \tau^\rho_{abc}(T_j, v_j) - \sum_{b+c=a} \tau^\rho_{(n-a)bc}(T_j, v_j) \right) = 0$$

where the first sum is over all components $T_1, T_2, \dots, T_u$ of $S - \lambda$, and where the second sum is over all three vertices of the triangle T_j.

Note that the equation associated with the index $n-a$ is, up to sign, the same as the equation associated with a. So in practice there are only $\lfloor \frac{n-1}{2} \rfloor$ equations here. One easily sees that these $\lfloor \frac{n-1}{2} \rfloor$ equations are linearly independent, as they involve different sets of terms $\tau^\rho_{a'b'c'}(T_j, v_j)$.

Proof. This is a consequence of Proposition 13.

We will use a slightly different notation for the formulas of Proposition 13. Write $g_k \to c_i^{\text{right}}$ to indicate that one end of the infinite leaf g_k spirals towards the right-hand side c_i^{right}. Similarly, we will write $(T_j, v_j) \to c_j^{\text{right}}$ when the triangle component T_j of $S - \lambda$ spirals towards c_i^{right}, in the direction of the vertex v_j of T_j. When $g_k \to c_i^{\text{right}}$, the quantity $\overline{\sigma}^\rho_a(g_k)$ denotes $\sigma^\rho_a(g_k)$ if the leaf g_k is oriented towards c_i and $\sigma^\rho_{n-a}(g_k)$ otherwise.

Proposition 13 computes the length $\ell^\rho_a(c_i)$ in terms of the triangle and shear invariants of the triangles and leaves spiraling on the right-hand side of c_i. The corresponding formula depends on the direction of the spiraling on the right-hand side of c_i. However, there is no such distinction to be made when computing the difference $\ell^\rho_a(c_i) - \ell^\rho_{n-a}(c_i)$. Indeed, independently of the direction of the spiraling,

$$\begin{aligned}\ell^\rho_a(c_i) - \ell^\rho_{n-a}(c_i) = &\sum_{g_k \to c_i^{\text{right}}} \left(\overline{\sigma}^\rho_a(g_k) - \overline{\sigma}^\rho_{n-a}(g_k) \right) \\ &+ \sum_{(T_j, v_j) \to c_i^{\text{right}}} \left(\sum_{b+c=n-a} \tau^\rho_{abc}(T_j, v_j) - \sum_{b+c=a} \tau^\rho_{(n-a)bc}(T_j, v_j) \right).\end{aligned}$$

Note that an infinite leaf g_k whose two ends spiral towards c_i^{right} will contribute two terms to the first sum, one for each end of g_k; there is a definite abuse of notation in this case, as these two contributions are both written as $\overline{\sigma}^\rho_a(g_k) - \overline{\sigma}^\rho_{n-a}(g_k)$, but are equal to $\sigma^\rho_a(g_k) - \sigma^\rho_{n-a}(g_k)$ for the positive end and $\sigma^\rho_{n-a}(g_k) - \sigma^\rho_a(g_k)$ for the negative end.

Switching attention to the left-hand side c_i^{left}, Proposition 13 similarly gives

$$\ell_a^\rho(c_i) - \ell_{n-a}^\rho(c_i) = -\sum_{g_k \to c_i^{\mathrm{left}}} \left(\overline{\sigma}_a^\rho(g_k) - \overline{\sigma}_{n-a}^\rho(g_k)\right) - \sum_{(T_j,v_j)\to c_i^{\mathrm{left}}} \left(\sum_{b+c=n-a} \tau_{abc}^\rho(T_j,v_j) - \sum_{b+c=a} \tau_{(n-a)bc}^\rho(T_j,v_j) \right).$$

Combining these two equations and summing over all closed leaves c_1, c_2, ..., c_s of the geodesic lamination λ, we obtain

$$\sum_{i=1}^{s} \sum_{(T_j,v_j)\to c_i} \left(\sum_{b+c=n-a} \tau_{abc}^\rho(T_j,v_j) - \sum_{b+c=a} \tau_{(n-a)bc}^\rho(T_j,v_j) \right) + \sum_{i=1}^{s} \sum_{g_k \to c_i} \left(\overline{\sigma}_a^\rho(g_k) - \overline{\sigma}_{n-a}^\rho(g_k)\right) = 0$$

where the statement $g_k \to c_i$ is shorthand for " $g_k \to c_i^{\mathrm{right}}$ or $g_k \to c_i^{\mathrm{left}}$ ", and similarly for $(T_j, v_j) \to c_i$.

Each infinite leaf g_k contributes two terms $\overline{\sigma}_a(g_k) - \overline{\sigma}_{n-a}(g_k)$ to the second sum, respectively equal to $\sigma_a^\rho(g_k) - \sigma_{n-a}^\rho(g_k)$ for the positive end of g_k and to $\sigma_{n-a}^\rho(g_k) - \sigma_a^\rho(g_k)$ for the negative end. It follows that all terms in this second sum cancel out, so that we are only left with the first sum.

A slightly different grouping of the terms of the first sum gives the equation of Proposition 30. □

A more conceptual and more general proof of Proposition 30, using the length functions of [Dr] and a cohomological argument, appears in [BoD].

We also prove in [BoD] that the relations of Proposition 30 are the only constraints satisfied by the triangle invariants $\tau_{abc}(T_j, v_j)$. This property could also be proved with the results and techniques of the current article, by elementary but somewhat cumbersome linear algebra. However, we prefer to omit it.

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